Data Structures and Algorithms

(CS210A)

Lecture 23

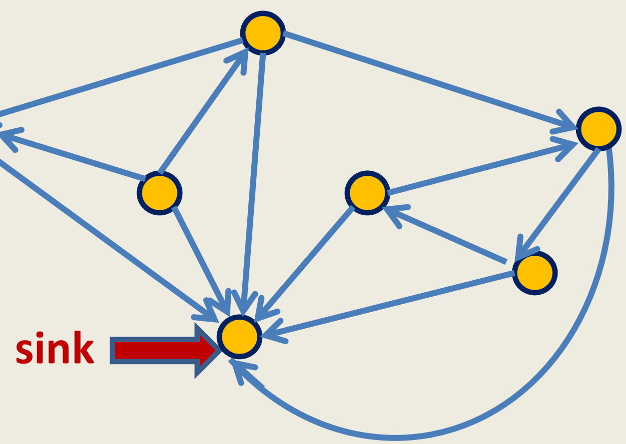
- Finding a **sink** in a directed graph
- Graph Traversal
  - Breadth First Search Traversal and its simple applications

An interesting problem

(Finding a sink)

Definition: A vertex ***x*** in a given directed graph is said to be a **sink** if

- There is no edge **emanating from** (leaving) ***x***
- Every other vertex has an edge **into** ***x***.



How many **sinks** can there be in ***G*** ?

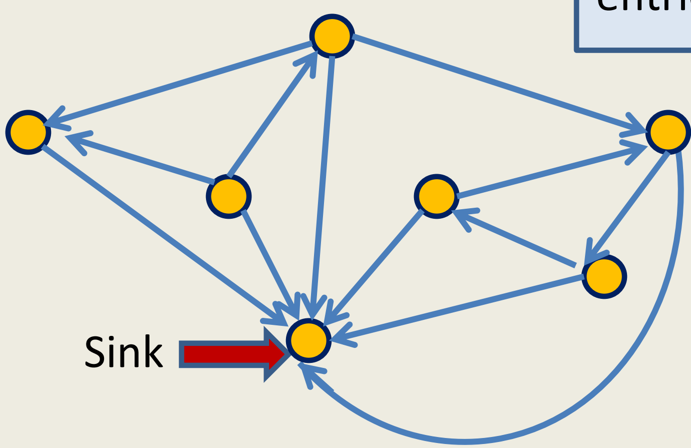
At most **1**.

An interesting problem

(Finding a sink)

Problem: Given a directed graph G=(V,E) in an **adjacency matrix** representation, design an **O**(n) time algorithm to determine if there is any **sink** in G.

We are allowed to look into only **O**(n) entries of the **Adjacency matrix M**.

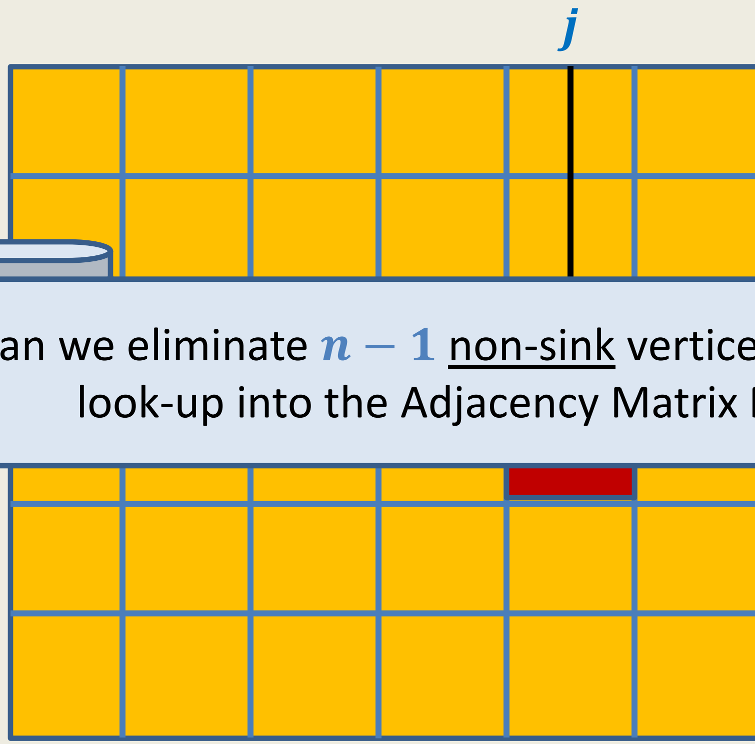


Question: Can we verify efficiently whether any given vertex i is a sink ?

Answer: Yes, in **O**(n) time only ☺

Look at ith **row** and ith **column** of **M**.

Key idea



Can we eliminate $n-1$ non-sink vertices in **O**(n) look-up into the Adjacency Matrix **M** ?

If **M**[i, j] = **0**, then j can not be sink

If **M**[i, j] = **1** , then i can not be sink

Algorithm to find a sink in a graph

Key ideas:

- Looking at a single entry in **M** allows us to discard one vertex from being a sink.
- It takes **O**(n) time to verify if a vertex i is a sink.

```
Find-Sink(M)          // M is the adjacency matrix of the given directed graph.
  s← 0;
  For(i=1 to n − 1)
  {
            If (M[s,i] =  ? )  ....?...;
  }
  Verify if s is a sink and output accordingly.
```

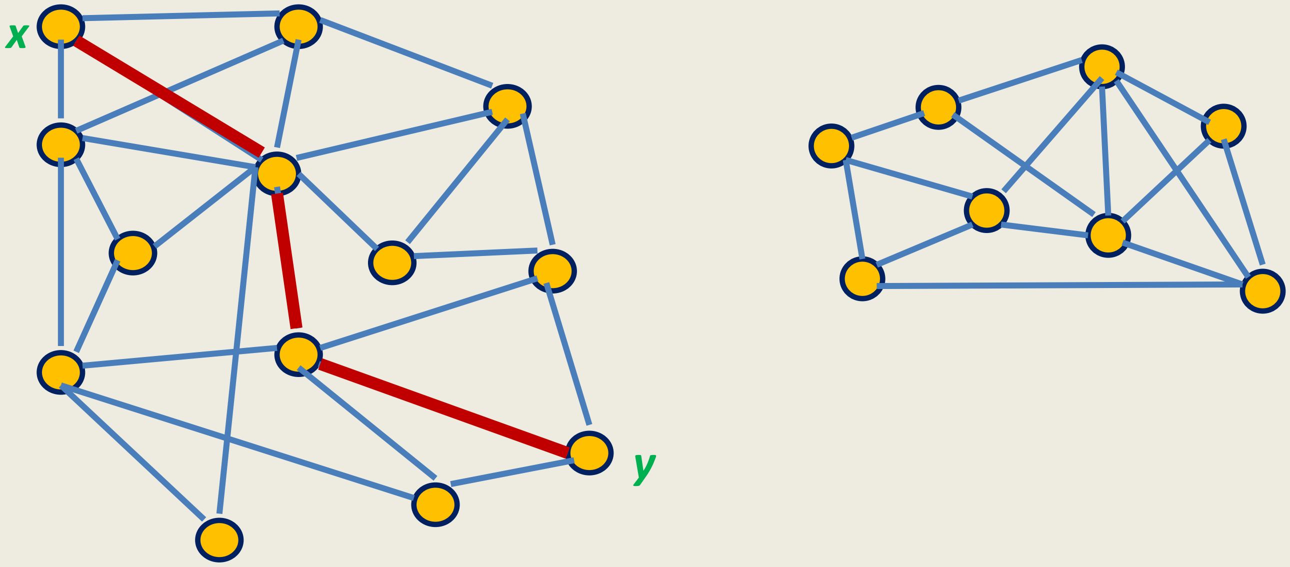
(Fill in the details of this pseudo code as a **Homework**.)

What is **Graph traversal** ?

Graph traversal

Definition:

A vertex ***y*** is said to be reachable from ***x*** if there is a **path** from ***x*** to ***y***.



Graph traversal from vertex *x*:

Starting from a given vertex ***x***, the aim is

to visit all vertices which are reachable from ***x***.

Non-triviality of graph traversal

- **Avoiding loop**:
  How to avoid visiting a vertex multiple times ?
  (keeping track of vertices already visited)

- **Finite number of steps** :
  The traversal **must stop** in finite number of steps.

- **Completeness** :
  We must visit **all** vertices reachable from the start vertex ***x***.

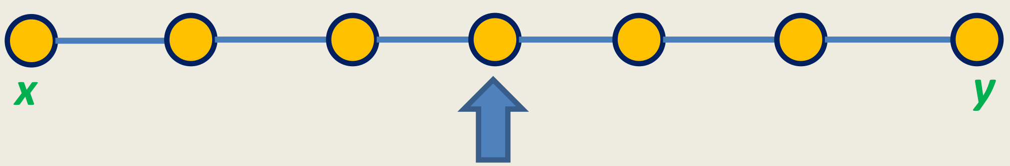
Breadth First Search traversal

We shall introduce this traversal technique through an interesting problem.

computing distances from a vertex.

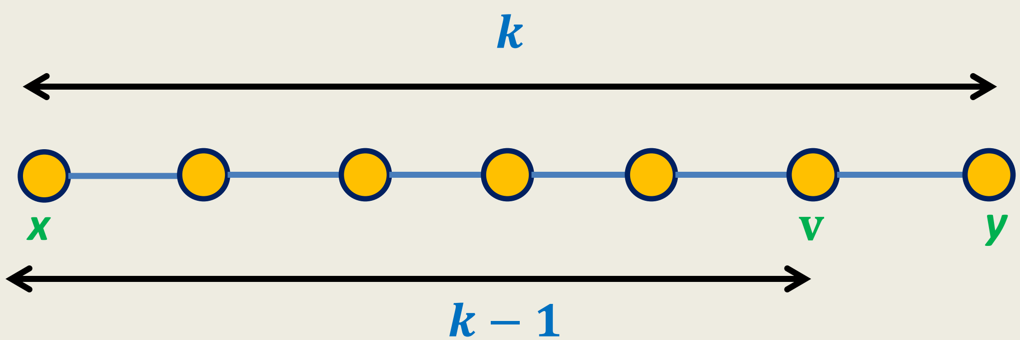
Notations and Observations

Length of a path: the **number of edges** on the path.



A path of length 6 between ***x*** and ***y***

Notations and Observations



Observation:

If <x, …,v,y> is a path of length k from x to y,

then what is the length of the path <x, …,v> ?

Answer: $k-1$

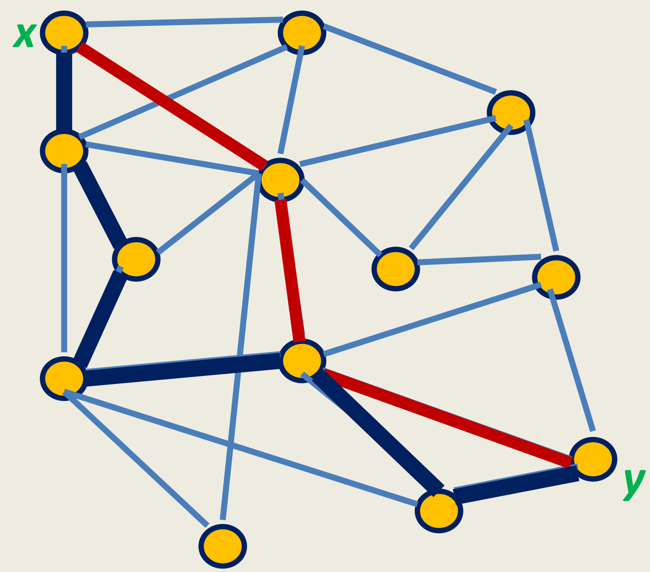
Question: What can be the maximum length of any path in a graph ?

Answer: $n-1$

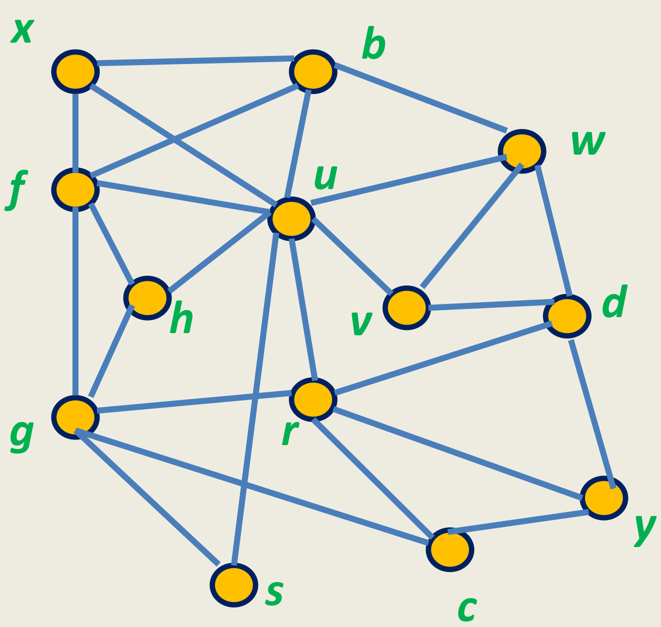
Notations and Observations

Shortest Path from *x* to *y*: A path from ***x*** to ***y*** of **<u>least</u>** <u>length</u>

Distance from ***x*** to ***y***: the **<u>length</u>** of the <u>shortest path</u> from ***x*** to ***y*****.**



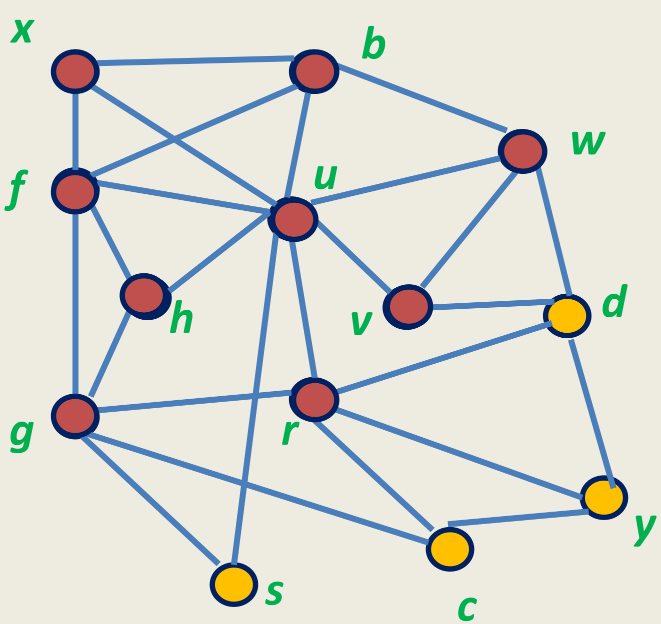
Shortest Paths
in Undirected Graphs



Problem:

How to compute distance to all vertices **reachable** from ***x*** in a given undirected graph ?

Shortest Paths
in Undirected Graphs



V_0 : Vertices at distance **0** from ***x***: {*x*}

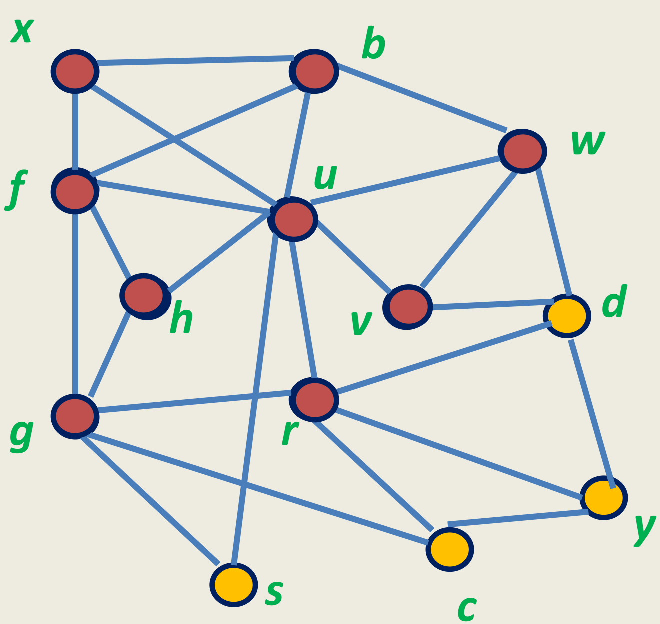
V_1 : Vertices at distance **1** from ***x***: {*f*,*u*,*b*}

V_2 : Vertices at distance **2** from ***x***: {*g*,*h*,*s*,*r*,*v*,*w*}

Why ?

While reporting V_2, you have (sub)consciously used an **important property** of shortest paths.
Can you state this property ?

Shortest Paths
in Undirected Graphs

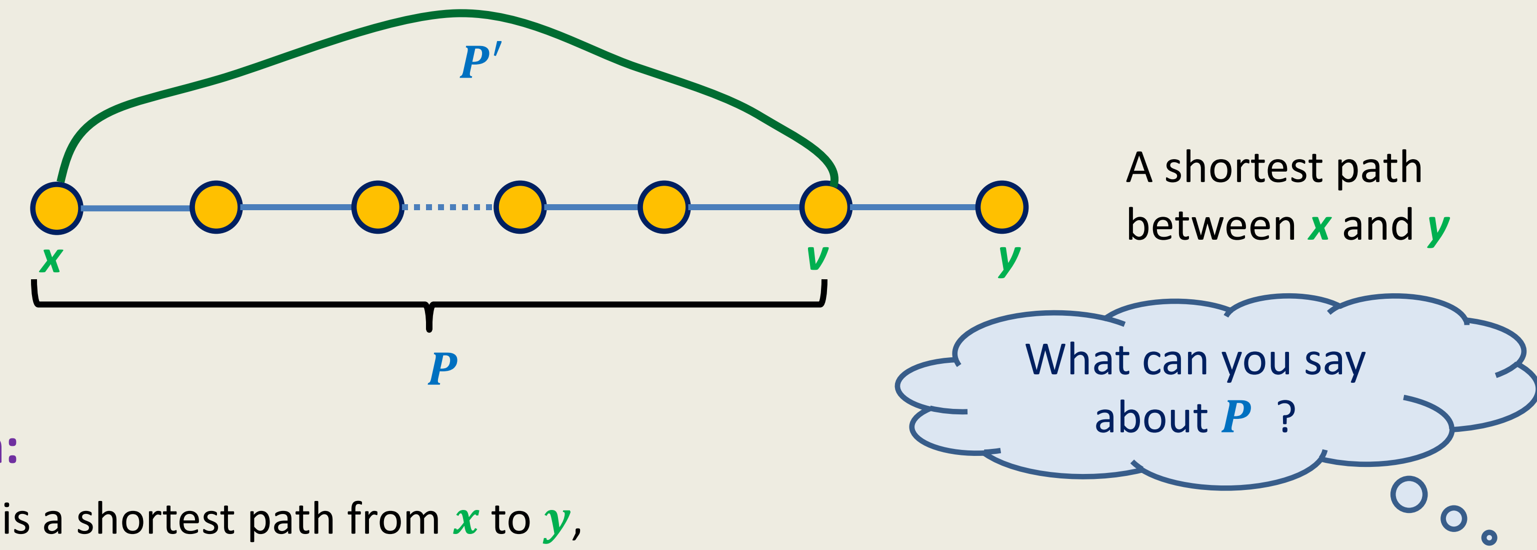


V_0 : Vertices at distance **0** from ***x***: {***x***}

V_1 : Vertices at distance **1** from ***x***: {***f***,***u***,***b***}

V_2 : Vertices at distance **2** from ***x***: {***g***,***h***,***s***,***r***,***v***,***w***}

An important property of shortest paths



Observation:

If <x, …,v,y> is a shortest path from x to y,

then <x, …,v> is also a shortest path.

Proof:

Suppose P = <x, …,v> is not a shortest path between x and v.

Then let P' be a shortest path between x and v.

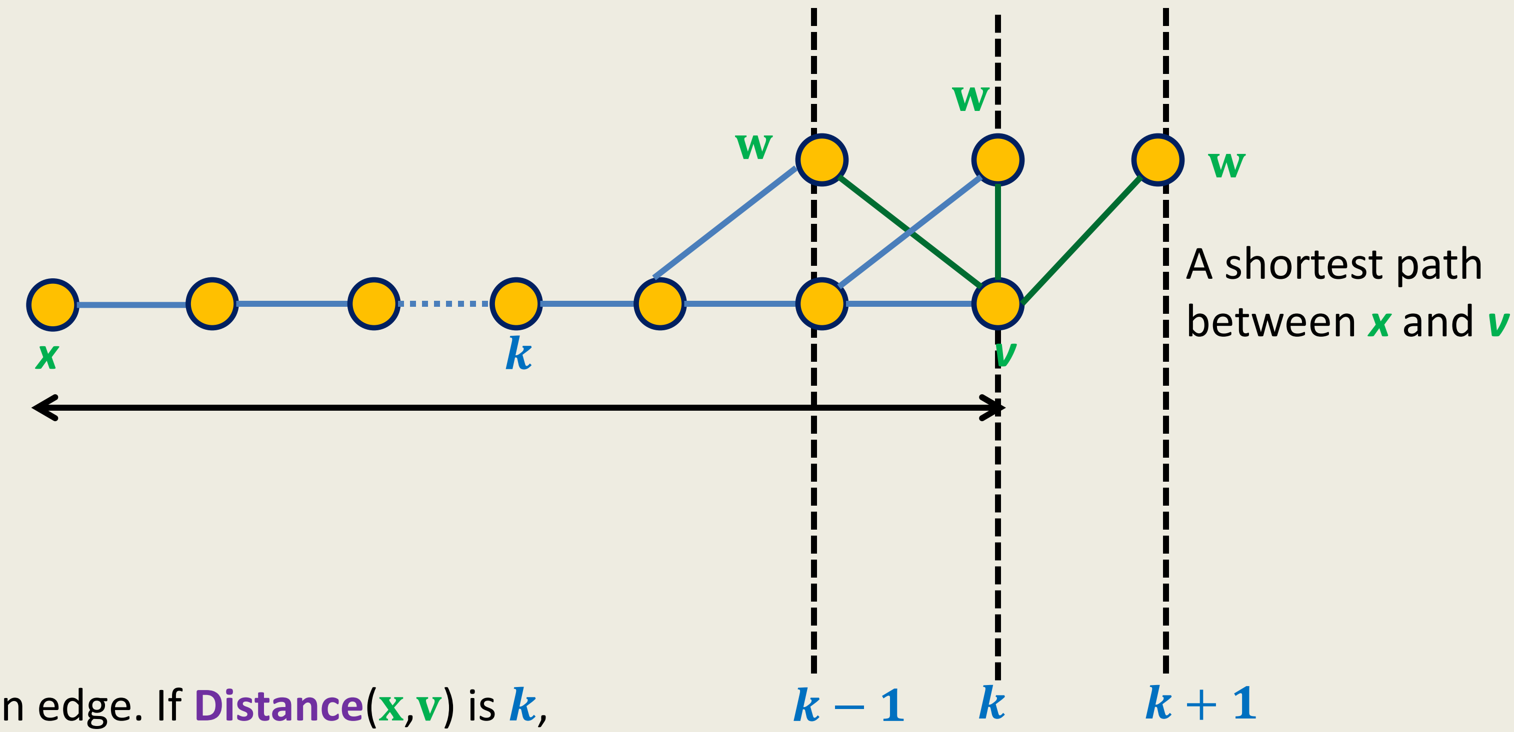
Length(P') < Length(P).

Question: What happens if we concatenate P' with edge (v, y) ?

Answer: a path between x and y shorter than the shortest-path <x, …,v,y> .

➔ Contradiction.

An important question



Question:

Let (**v**,**w**) be an edge. If **Distance**(**x**,**v**) is $\boldsymbol{k}$,

then what can be **Distance**(**x**,**w**) ?

Answer: an element from the set $\{\boldsymbol{k-1}, \boldsymbol{k}, \boldsymbol{k+1}\}$ only.

Relationship among vertices at different distances from *x*

V_0 : Vertices at distance **0** from ***x*** = {***x***}

V_1 : Vertices at distance **1** from ***x*** =

Neighbors of V_0

V_2 : Vertices at distance **2** from ***x*** =

Those Neighbors of V_1 which **do not** belong to V_0 or V_1

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.

.

V_{i+1} : Vertices at distance **i+1** from ***x*** =

Those Neighbors of V_i which **do not** belong to $V_i - 1$ or V_i

How to distinguish the neighbors of V_i which belong to V_{i+1} from those which belong to V_j, $j \leq i$?

How can we compute V_{i+1} ?

Key idea: compute V_i's in increasing order of i.

Initialize **Distance**[**v**] ← ∞ of each vertex **v** in the graph.

Initialize **Distance**[**x**] ← 0.

- First compute V_0 .
- Then compute V_1 .
- …
- Once we have computed V_i , for every neighbor **v** of a vertex in V_i,

  If **v** is in V_j for some $j \in \{i, i-1\}$, then **Distance**[**v**] = a number ≤ i

  If **v** is in V_{i+1}, **Distance**[**v**] = ∞

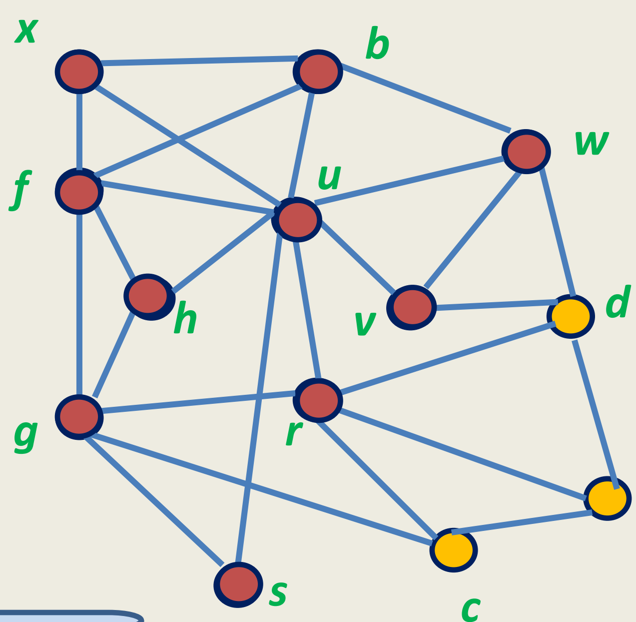
We can thus distinguish the neighbors of V_i which belong to V_{i+1} from those which belong to V_j.

A neat algorithm for computing distances from *x*

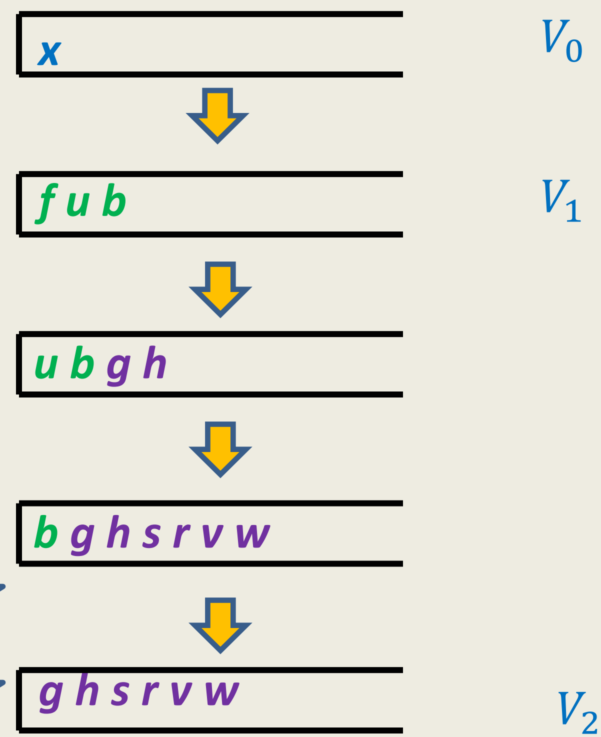
we need an algorithm which traverses/visits the vertices in **non-decreasing** order of **distances** from *x*

This traversal algorithm is called **BFS** (breadth first search) traversal

Using a queue for traversing vertices in non-decreasing order of distances



Compute distance of vertices from *x*:



Remove *x* and for each neighbor of *x* that

Remove *b* and for each neighbor of *b* that was unvisited, mark it visited and put it into queue.

BFS traversal from a vertex

```
BFS(G, x)
  CreateEmptyQueue(Q);
  Distance(x) ← 0;
  Enqueue(x,Q);
  While(  Not IsEmptyQueue(Q)  )
  {       v ←Dequeue(Q);
          For each neighbor w of v
          {
                  if (Distance(w) = ∞)
                  {  Distance(w) ←  Distance(v) +1  ;
                     Enqueue(w, Q);  ;
                  }
          }
  }
```

Running time of BFS traversal

```
BFS(G, x)
    CreateEmptyQueue(Q);
    Distance(x) ← 0;
    Enqueue(x,Q);
    While( Not IsEmptyQueue(Q) )
    {       v ← Dequeue(Q);
            For each neighbor w of v
            {
                    if (Distance(w) = ∞)
                    {   Distance(w) ← Distance(v) +1 ;
                        Enqueue(w, Q); ;
                    }
            }
    }
```

O(deg(v)) — for the body of the For loop

> A vertex can enter queue **at most** once.
> Prove this claim first.

Running time of BFS(x) = no. of edges in **the connected component** of **x**.

Correctness of BFS traversal

Question: What do we mean by correctness of **BFS** traversal from vertex **x** ?

Answer:

- **All vertices** reachable from **x** **get visited**.
- Vertices get visited in the **non-decreasing order of their distances** from **x**.
- At the end of the algorithm,

  Distance(**v**) **is the distance** of vertex **v** from **x**.

We shall discuss this proof in the next class. Please think over it for some time **before coming to the class.**

A useful advice

An effective way to master the technique of **proving correctness of an algorithm** is to

Do each home **work exercise**
(about proof of correctness)
that is asked in the class <u>before</u> attending the next class.

There is no escape. There will be question on proof of correctness in the next course. ☺