# Data Structures and Algorithms

(CS210A)

**Lecture 17:**

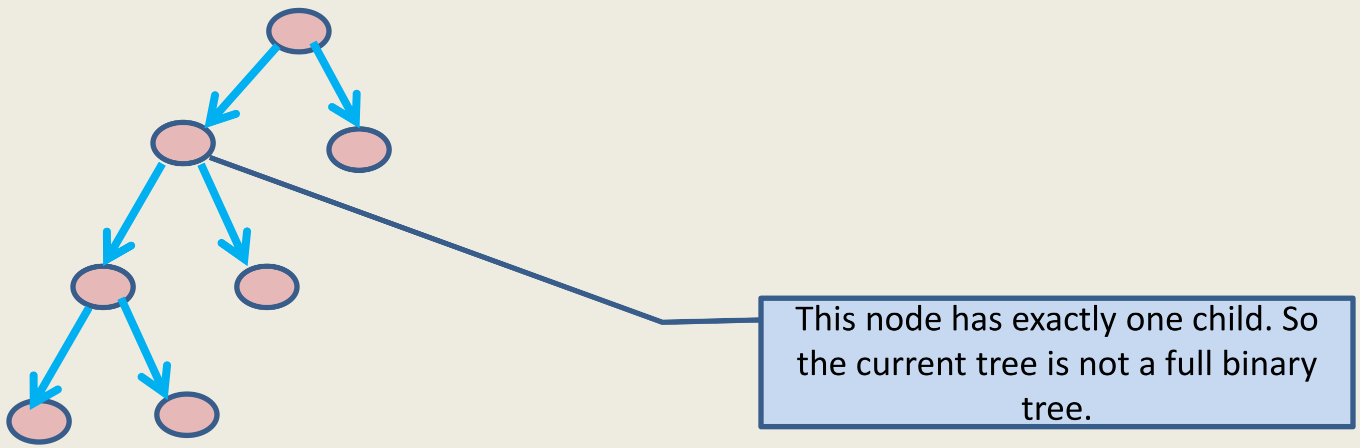
**Height balanced BST**

- Red-black trees

# Terminologies

**Full binary tree:**

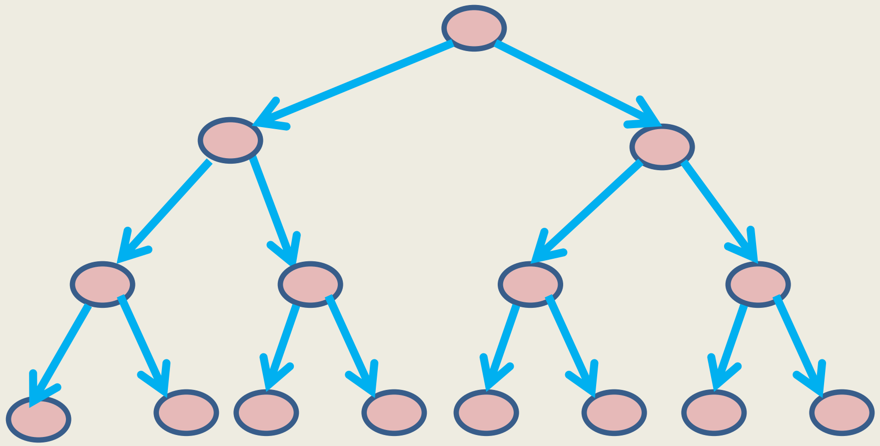
A binary tree where every internal node has <u>exactly two children.</u>



# Terminologies

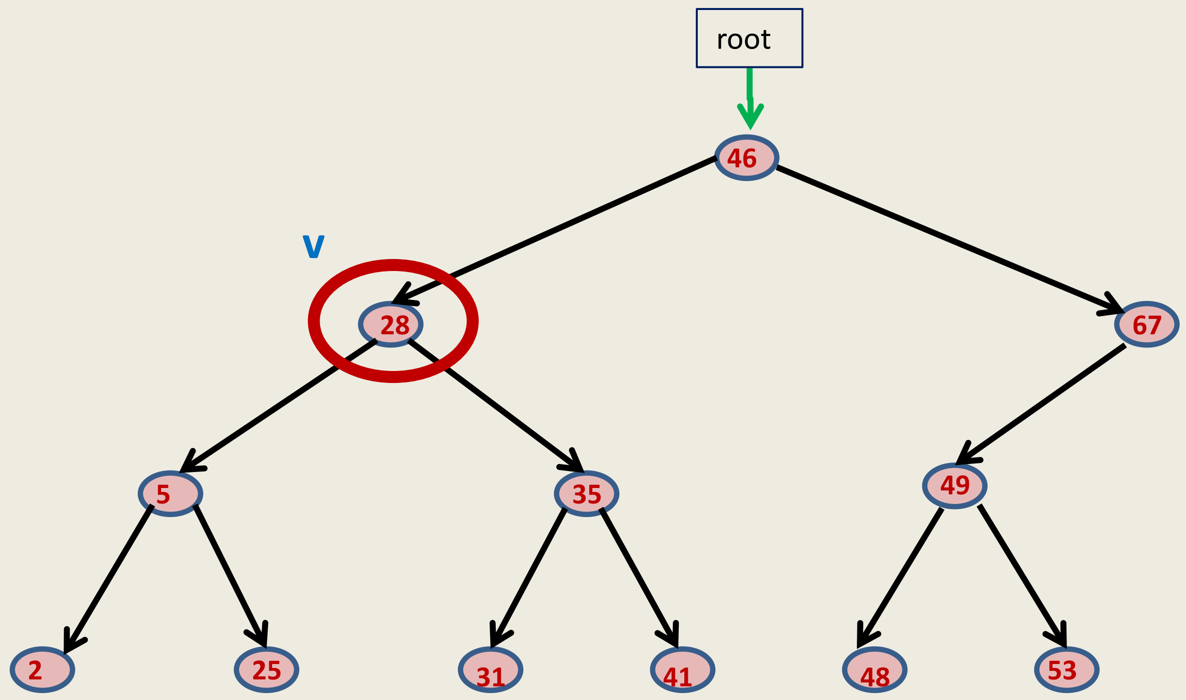
**Complete binary tree:**

A full binary tree where every leaf node is at the **same level**.



We shall later extend this definition when we discuss "**Binary heap**".

# Binary Search Tree

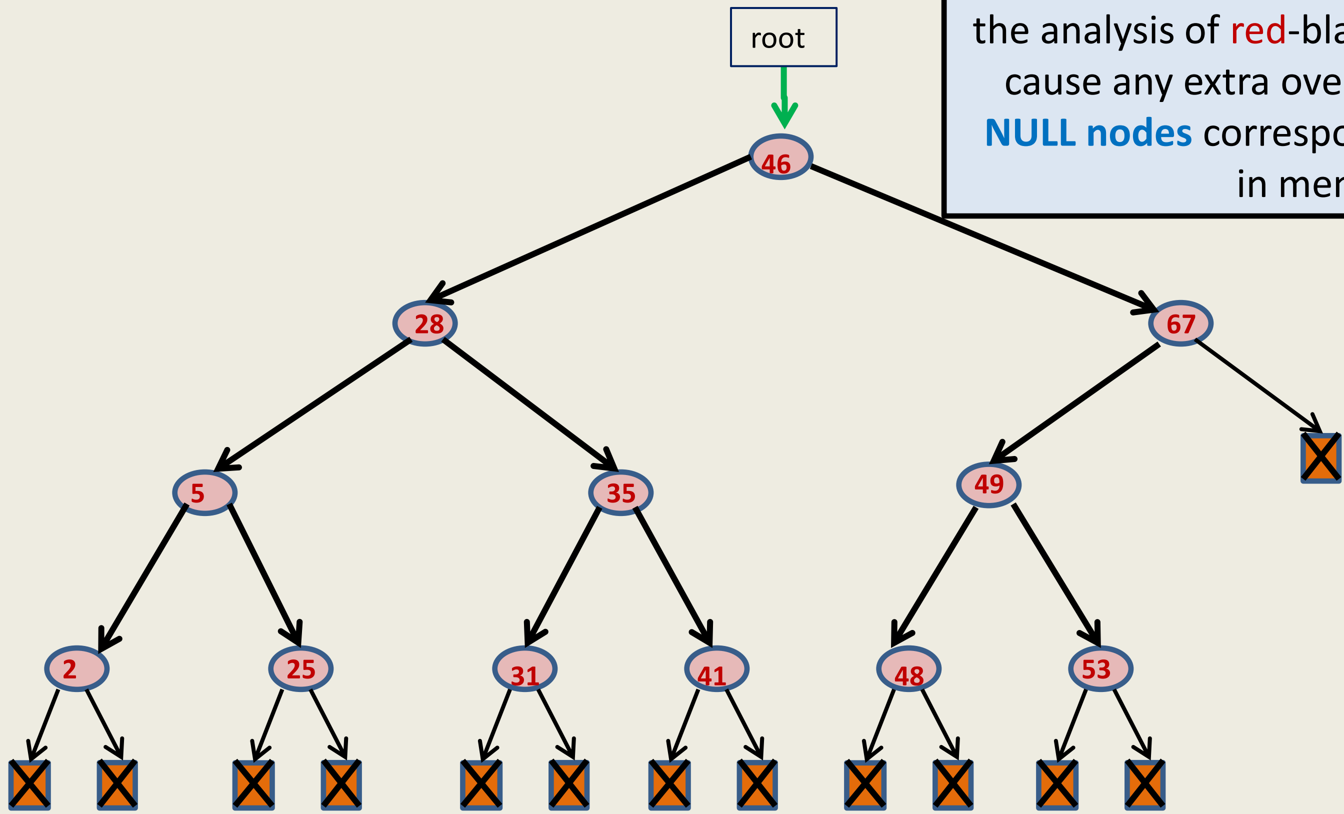



**Definition:** A Binary Tree **T** storing values is said to be Binary Search Tree if for each node **v** in T

- If **left**(**v**) <> NULL, then **value**(**v**) > **value** of every node in **subtree**(**left**(**v**)).
- If **right**(**v**)<>NULL, then **value**(**v**) < **value** of every node in **subtree(right**(**v**)).

# Binary Search Tree: a slight change

This transformation is merely to help us in the analysis of red-black trees. It does not cause any extra overhead of space. All **NULL nodes** correspond to a <u>single node</u> in memory.




Henceforth, for each **NULL child link** of a node in a **BST**, we create a **NULL** node

➔ **1.** Each **leaf node** in a BST will be a **NULL** node.

**2.** the BST will always be a **full binary tree**.

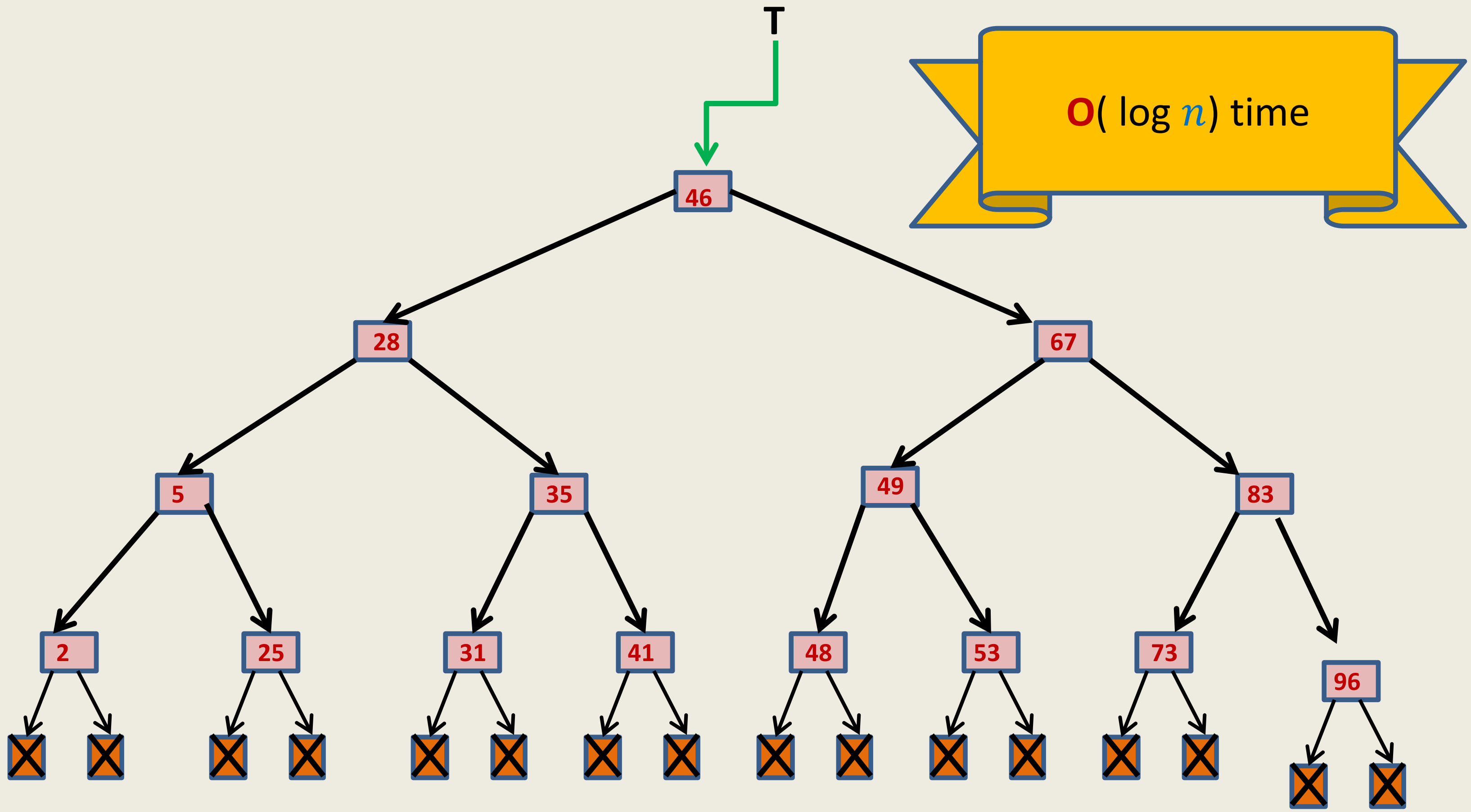
# A fact we noticed in our previous discussion on BSTs (Lecture 9)

Time complexity of Search(T,x) and Insert(T,x) in a Binary Search Tree T = **O(Height(T))**

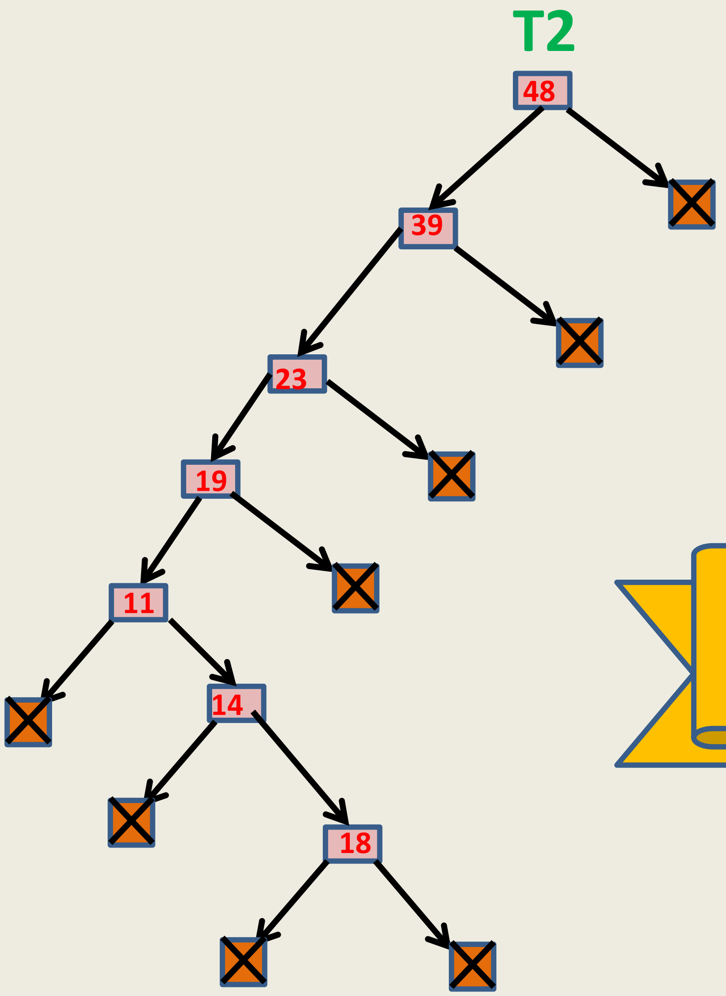
**Height**(*T*):

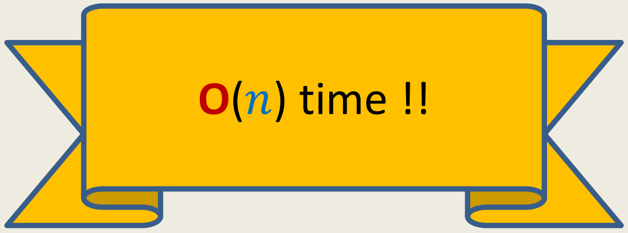
The maximum number of nodes on any path from root to a leaf node.

# Searching and inserting in a perfectly balanced BST



# Searching and inserting in a skewed BST on $n$ nodes



# Nearly balanced Binary Search Tree

**Terminology:**

**size** of a binary tree is the number of nodes present in it.

**Definition:** A binary search tree **T** is said to be **nearly balanced** at node **v**, if

**size**(**left**(**v**)) $\leq \frac{3}{4}$ **size**(**v**)

and

**size**(**right**(**v**)) $\leq \frac{3}{4}$ **size**(**v**)

**Definition:** A binary search tree **T** is said to be **nearly balanced** if it is **nearly balanced** at each node.

# Nearly balanced Binary Search Tree

- **Search**(T,x) operation is the same.

- Modify **Insert**(T,x) operation as follows:
  - Carry out normal insert and update the **size** fields of nodes traversed.
  - If **BST** T is ceases to be **nearly imbalanced** at any node **v**,
    transform **subtree(v)** into **perfectly balanced BST**.

➔ **O**(log $n$) time for **search**

➔ **O**($n$ log $n$) time for $n$ **insertions**

This fact will be proved soon in the next to the next class.

**Disadvantages:**

- How to handle **deletions** ?
- Some insertions may take **O**($n$) time ☹

# Can we achieve O(log $n$) time for search/insert/delete ?

- AVL Trees [1962]
- Red Black Trees [1978]

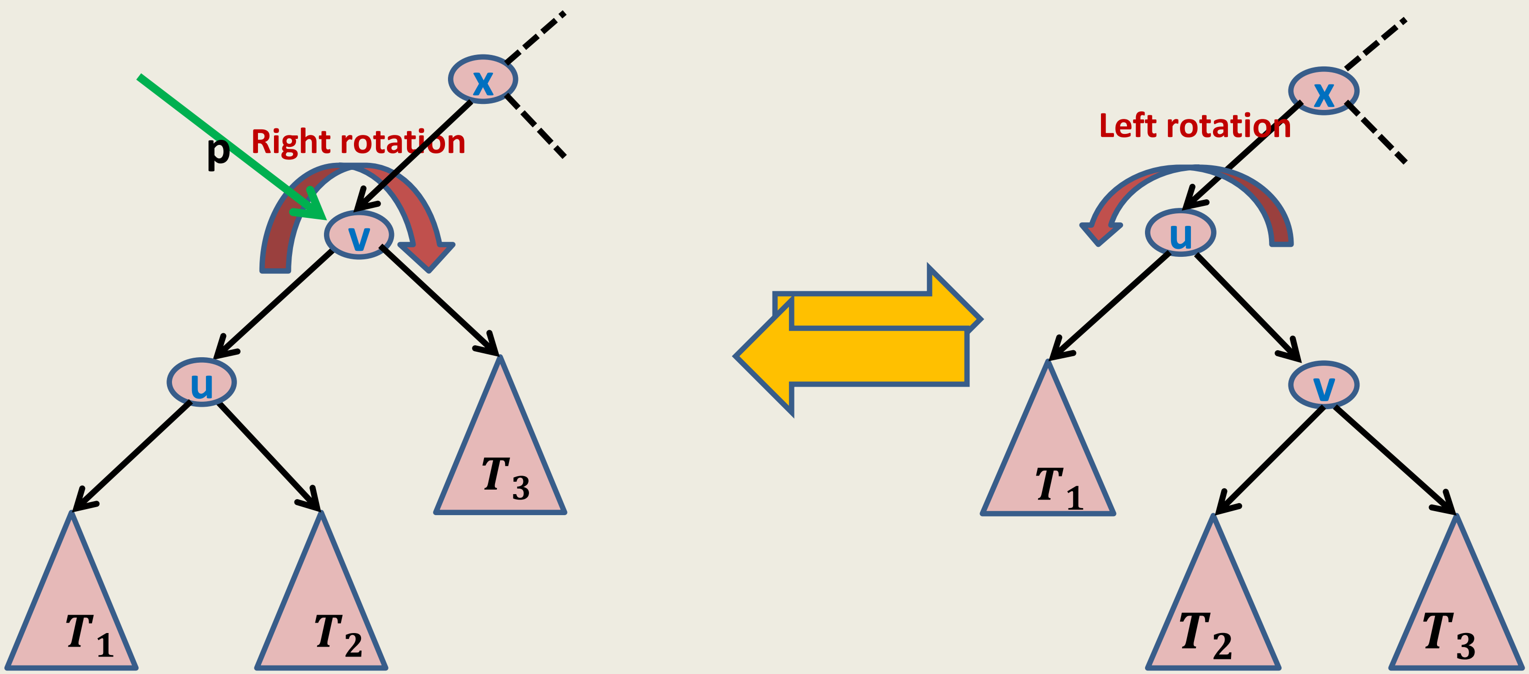
# **Rotation** around a node

**An important tool for balancing trees**

Each height balanced **BST** employs this tool which is derived from the **flexibility** which is *hidden* in the structure of a **BST**.

This flexibility (**pointer manipulation**) was inherited from linked list ☺.

# Rotation around a node




*Note that the tree* ***T*** *continues to remain a BST even after rotation around any node.*

# Red Black Tree
## A height balanced BST

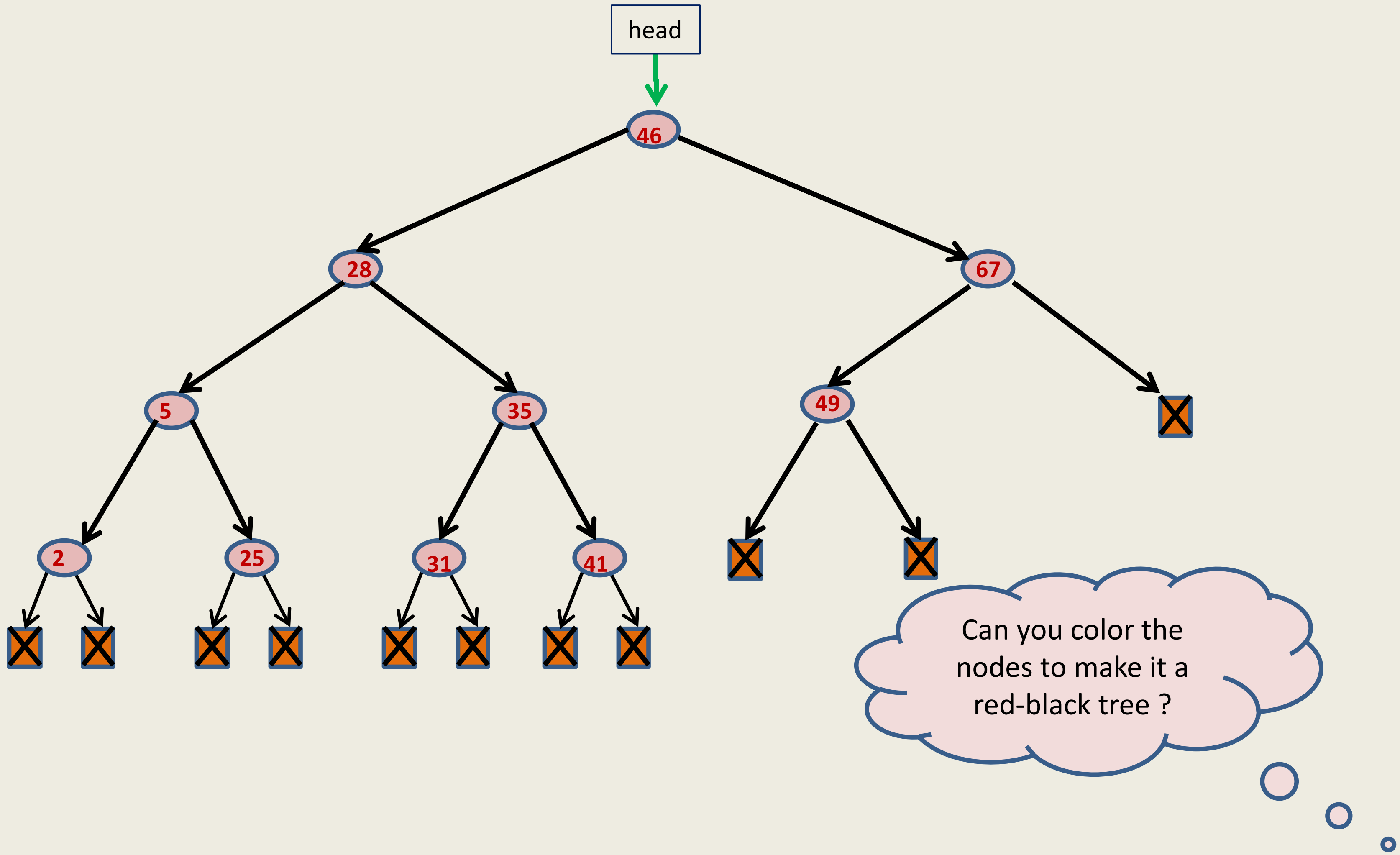
# Red **Black** Tree

Red-Black tree is a binary search tree
satisfying the following properties:

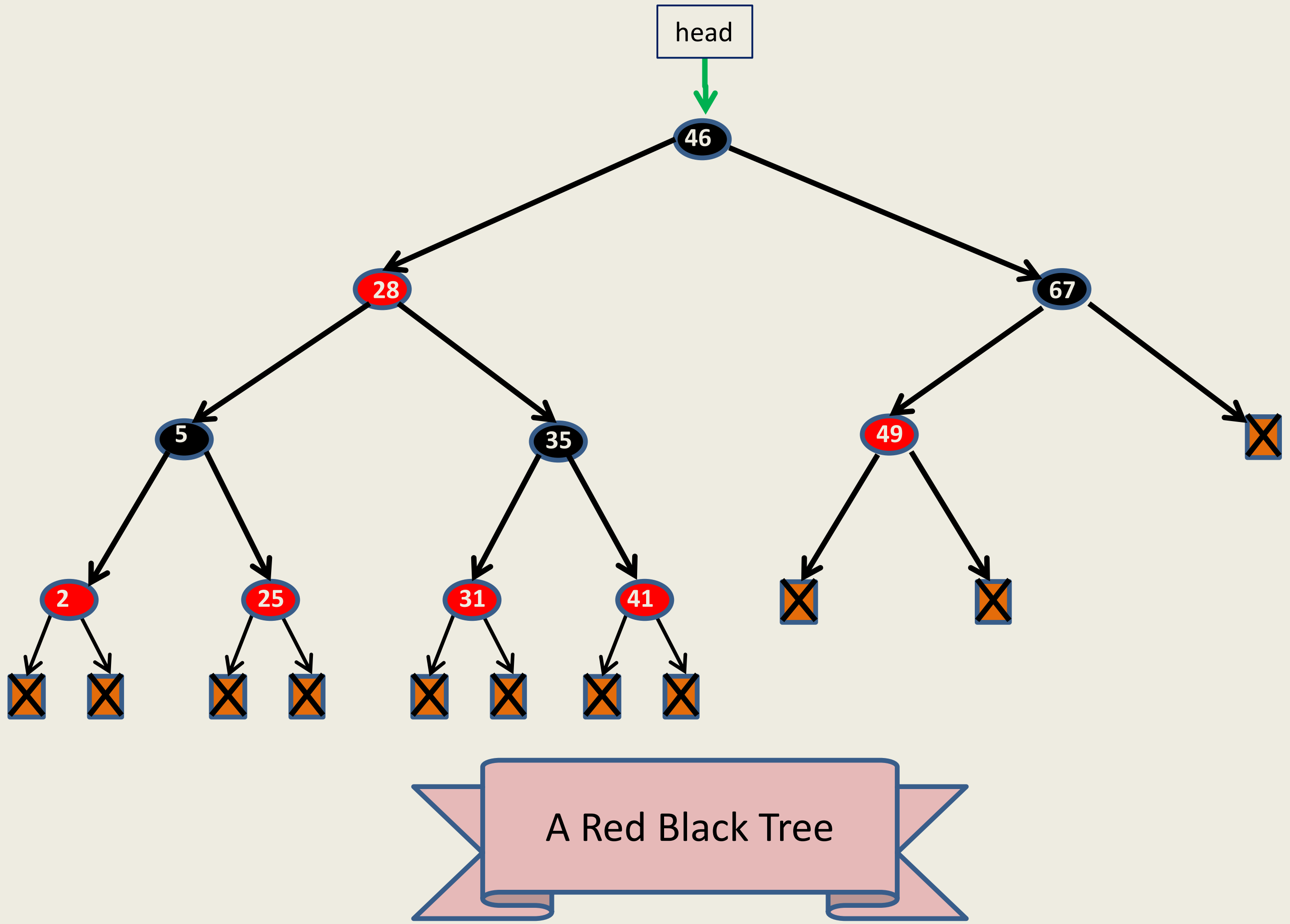
- Each node is colored **red** or **black.**
- Each leaf is colored **black** and so is the root.
- Every **red** node will have both its children **black.**
- No. of <u>**black** nodes</u> on a path from root to each leaf node is same.

**black** height

# A binary search tree



# A binary search tree



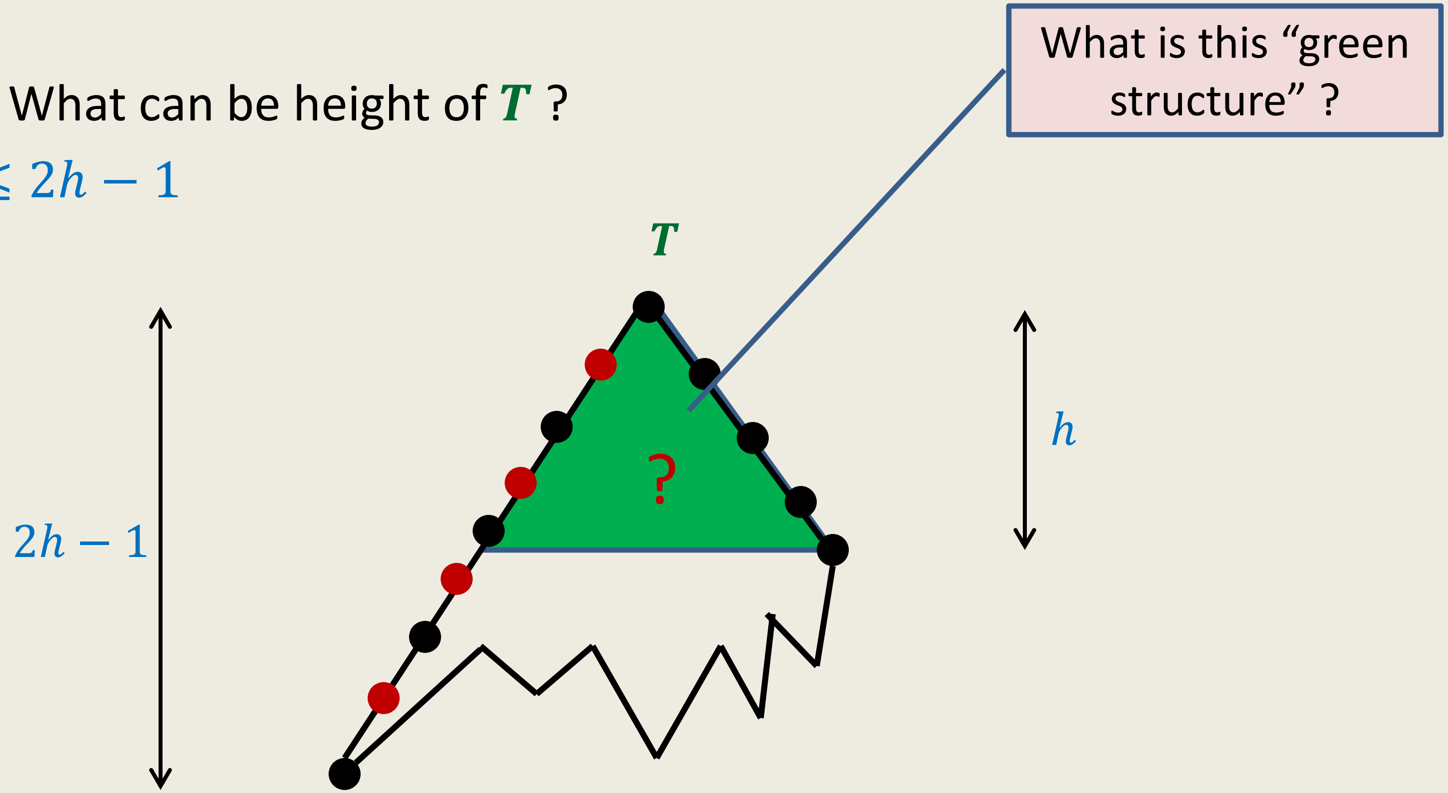
# Why is a red black tree height balanced ?

$\boldsymbol{T}$ : a **red black tree**

$h$ : **black** height of $\boldsymbol{T}$.

**Question**: What can be height of $\boldsymbol{T}$ ?

**Answer**: $\leq 2h - 1$




**Homework**: Ponder over the above hint to prove that $\boldsymbol{T}$ has $\geq 2^h - 1$ elements.

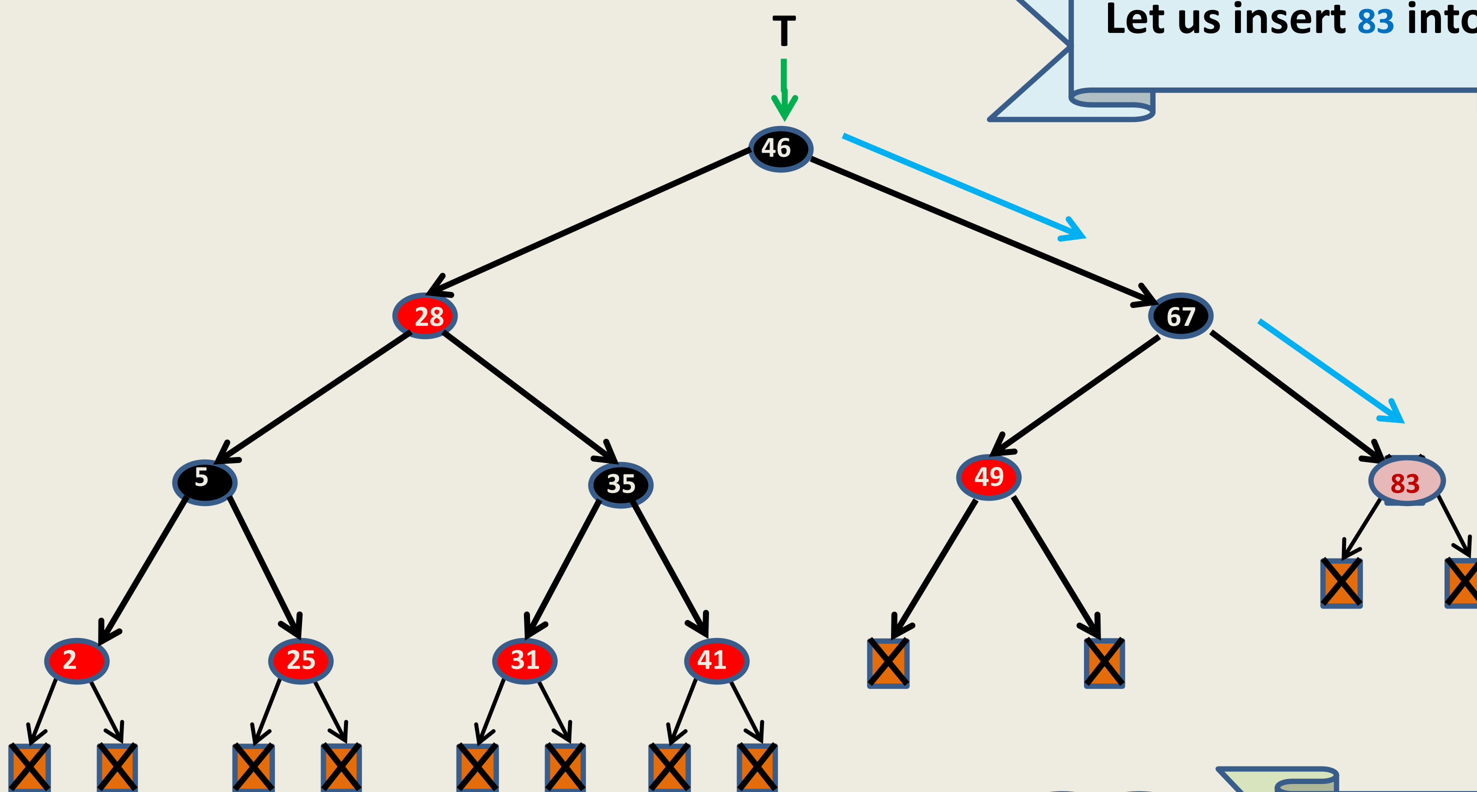
# Insertion in a Red Black tree

All it involves is

- playing with **colors** ☺
- and **rotations** ☺

# Insertion in a red-black tree

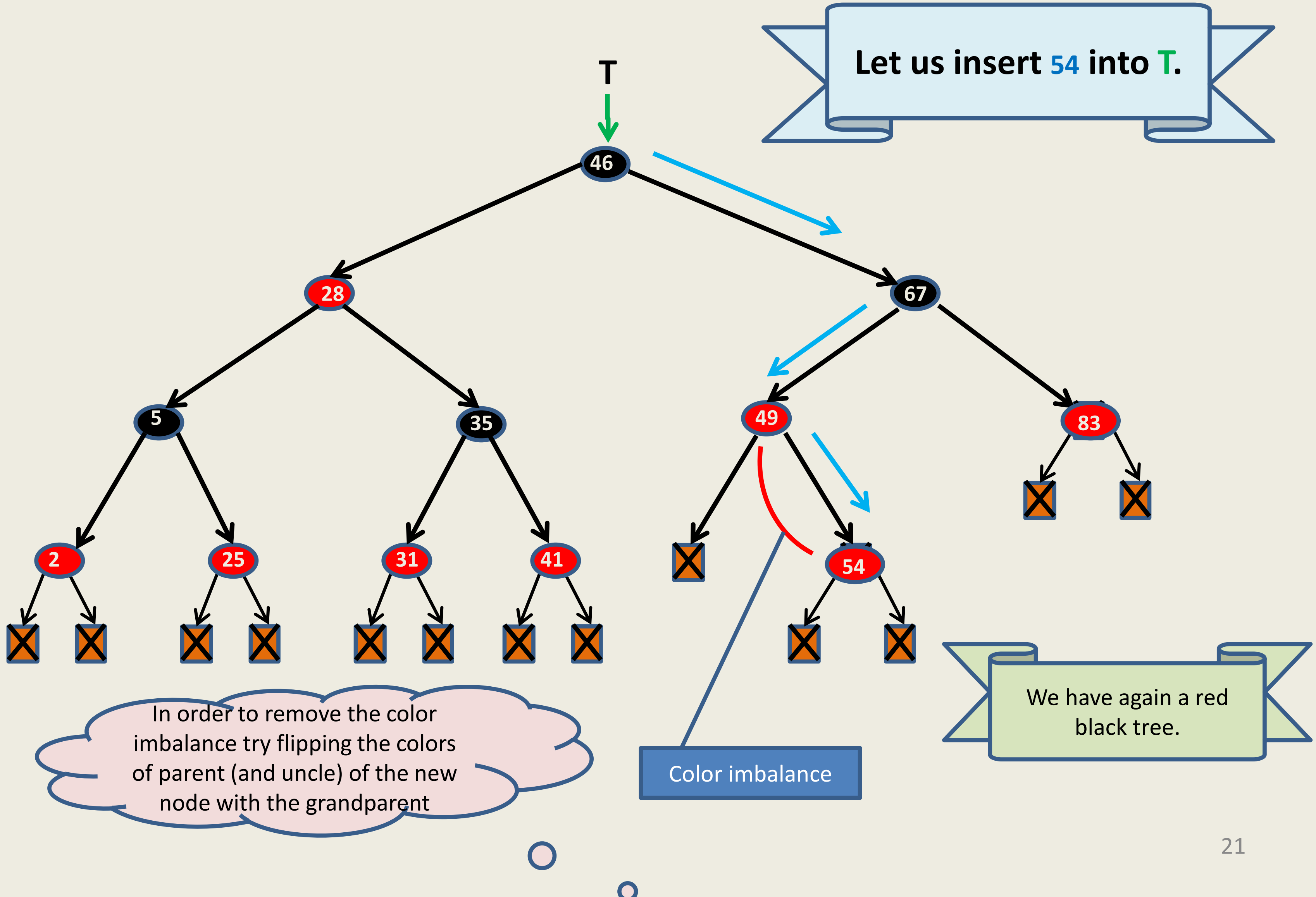
Let us insert 83 into T.



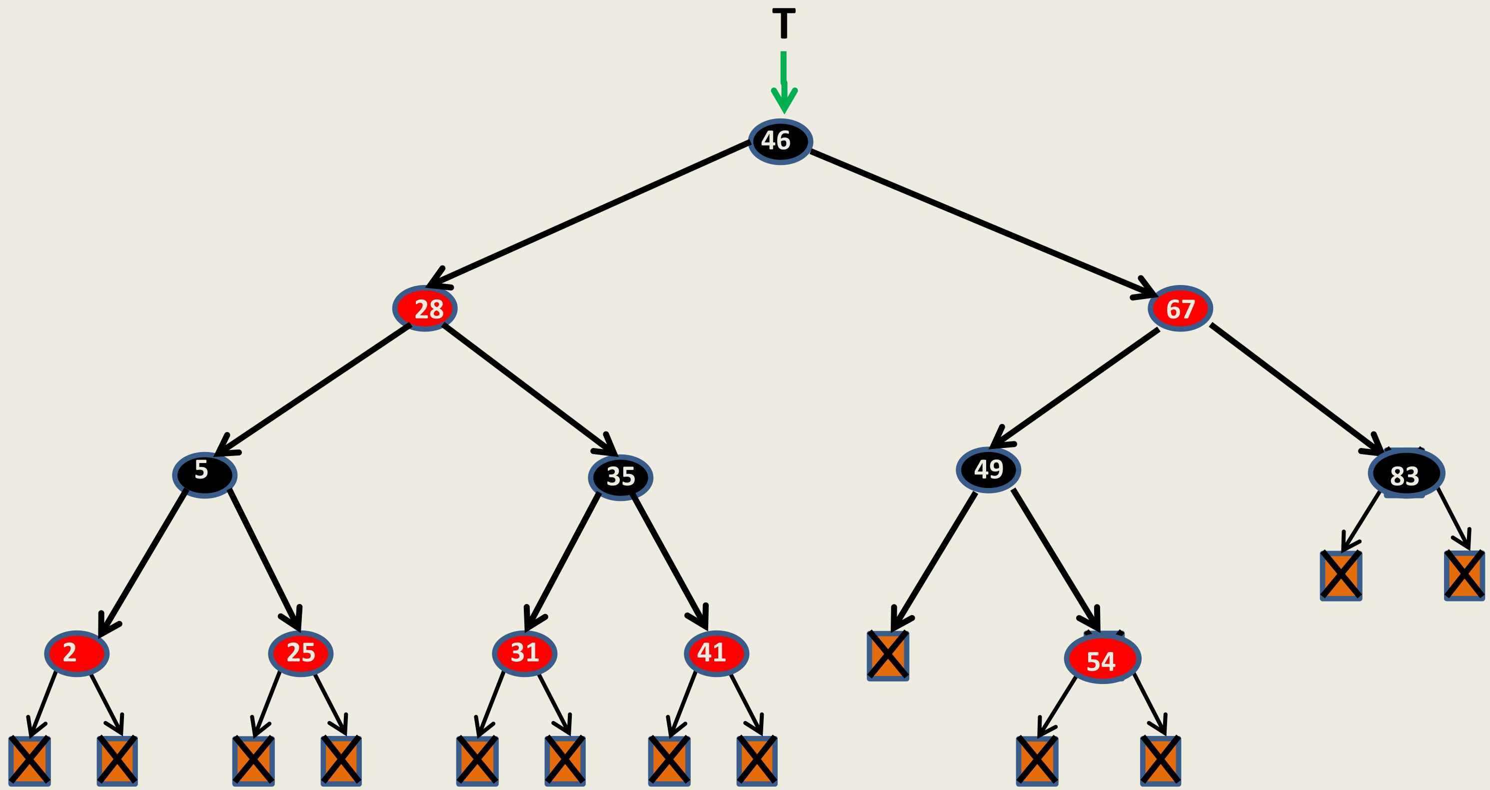

What color should we assign to the new node ?

We shall assign every newly inserted node a red color. (give reason)

# Insertion in a red-black tree

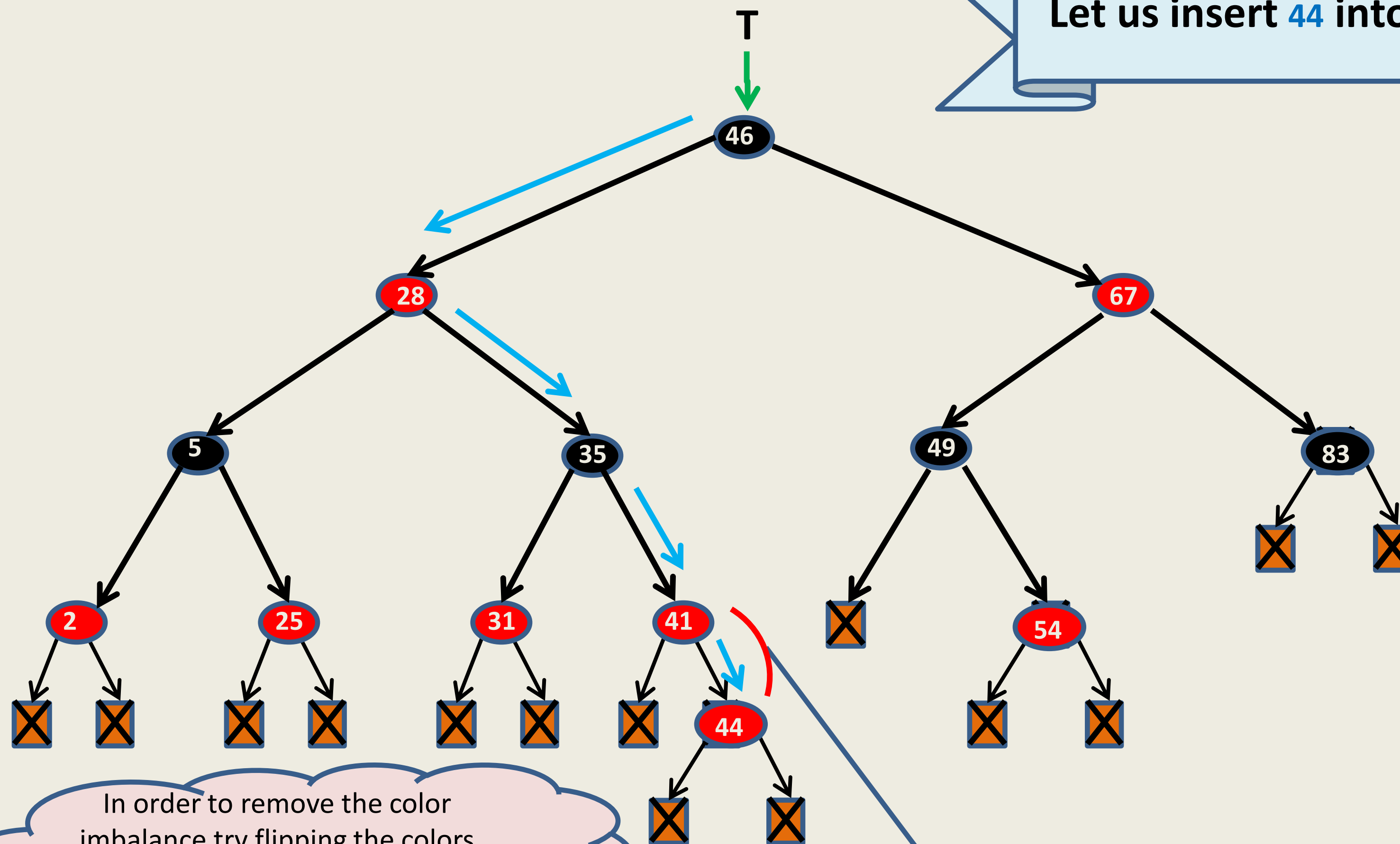


# Insertion in a red-black tree



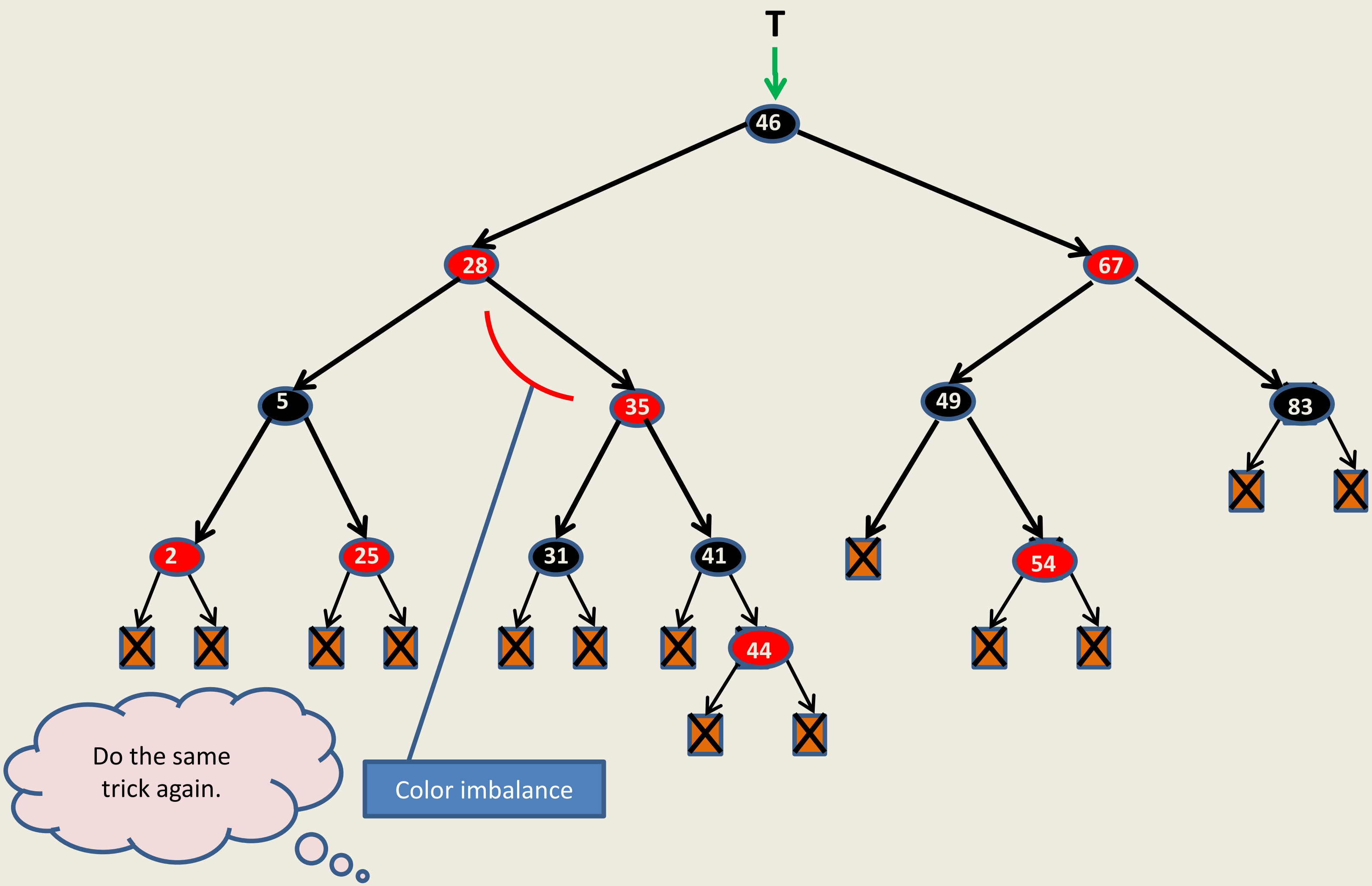
# Insertion in a red-black tree

Let us insert 44 into T.

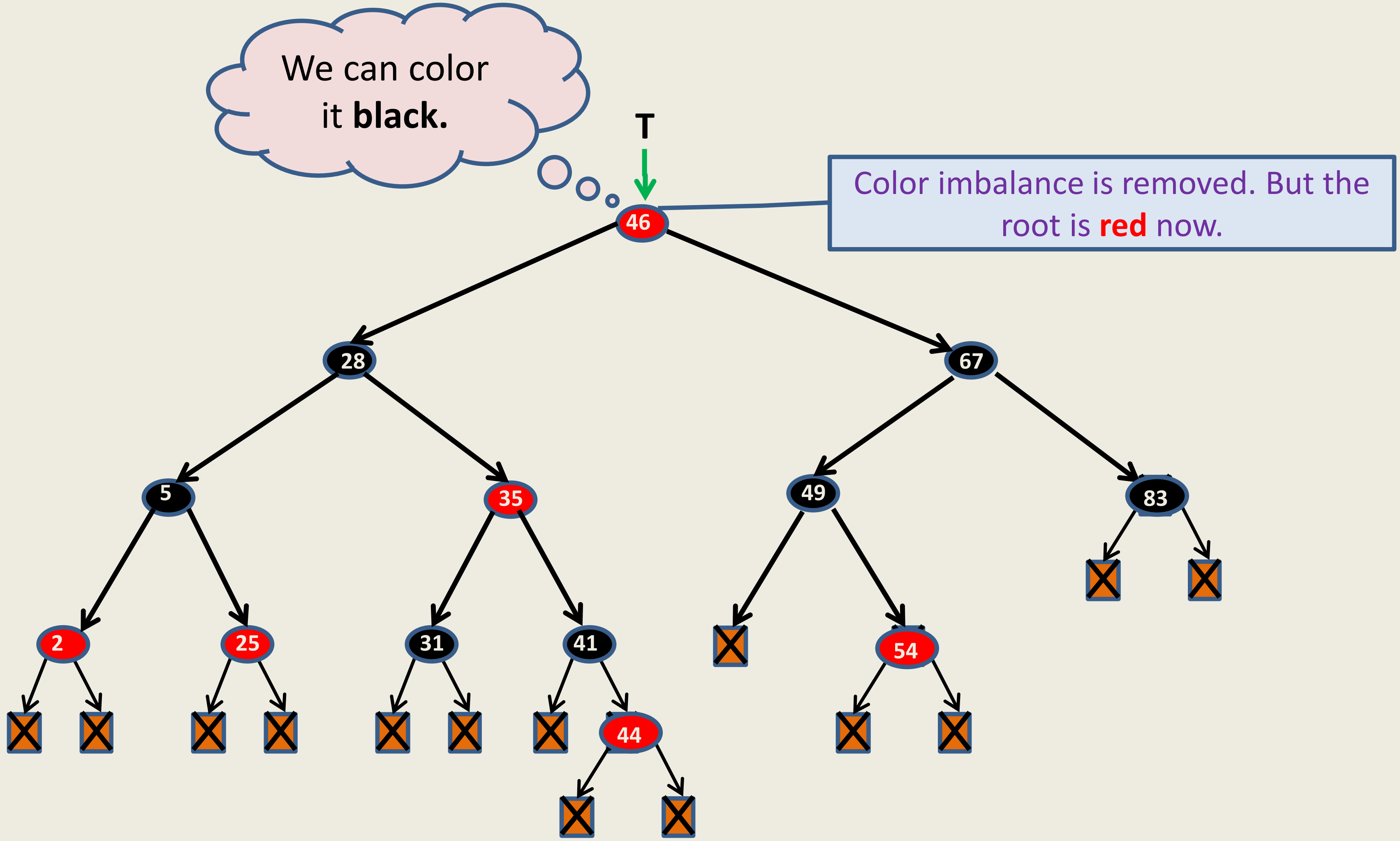



In order to remove the color imbalance try flipping the colors of parent (and uncle) of the new node with the grandparent

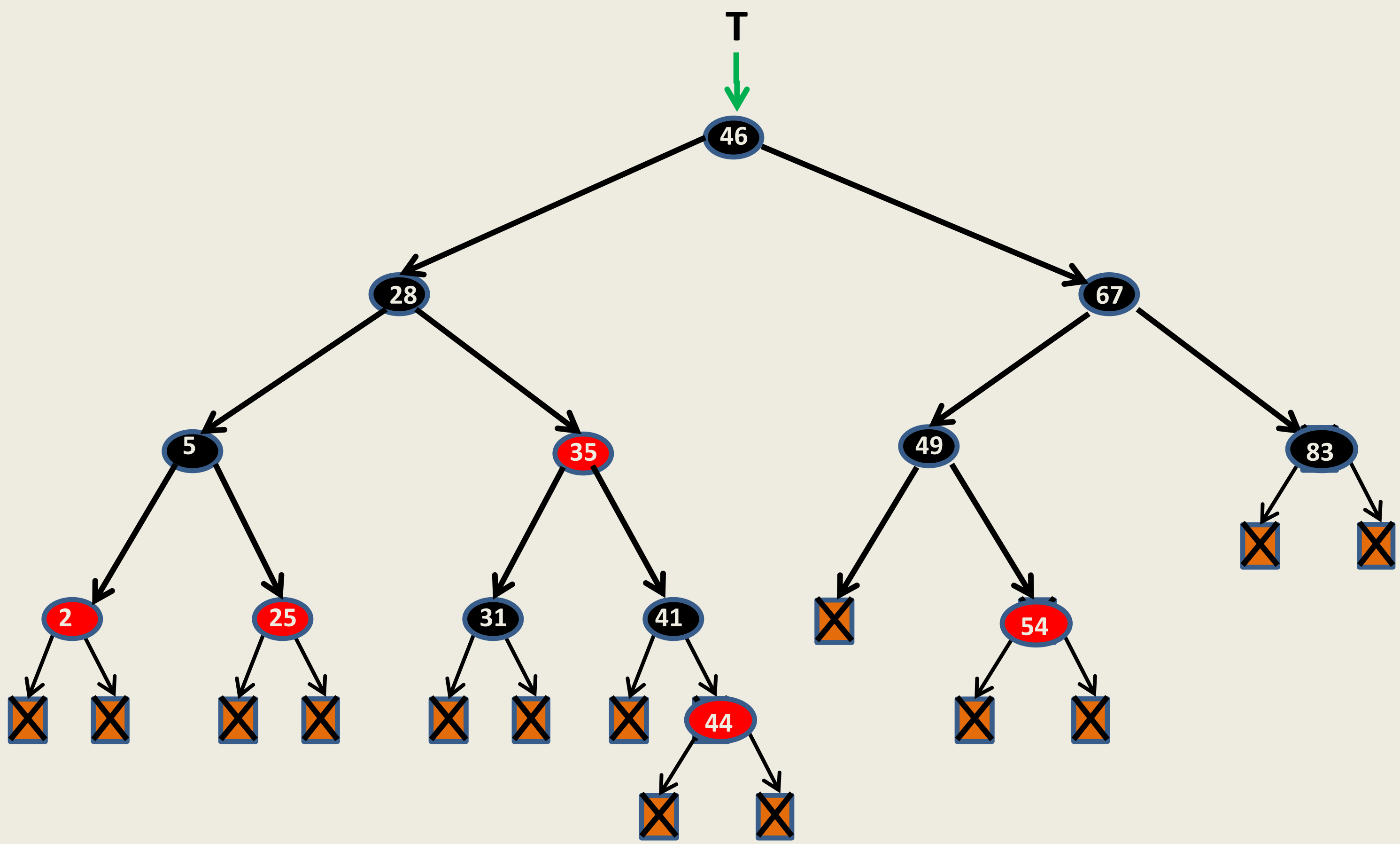
# Insertion in a red-black tree



# Insertion in a red-black tree



# Insertion in a red-black tree



# Insertion in a red-black tree

## summary till now …

Let $p$ be the newly inserted node. Assign **red** color to $p$.

**Case 1:** ***parent***($p$) is **black**

nothing needs to be done.

**Case 2:** ***parent***($p$) is **red** and ***uncle***($p$) is **red**,

Swap colors of ***parent*** (and ***uncle***) with ***grandparent***($p$).

This balances the color at $p$ but may lead to imbalance of color at grandparent of $p$. So $p \leftarrow$ ***grandparent***($p$), and proceed upwards similarly.

If in this manner $p$ becomes **root**, then we color it **black**.

**Case 3:** ***parent***($p$) is **red** and ***uncle***($p$) is **black**.

**This is a nontrivial case. So we need some more tools ….**

# Handling case 3

# Description of Case 3

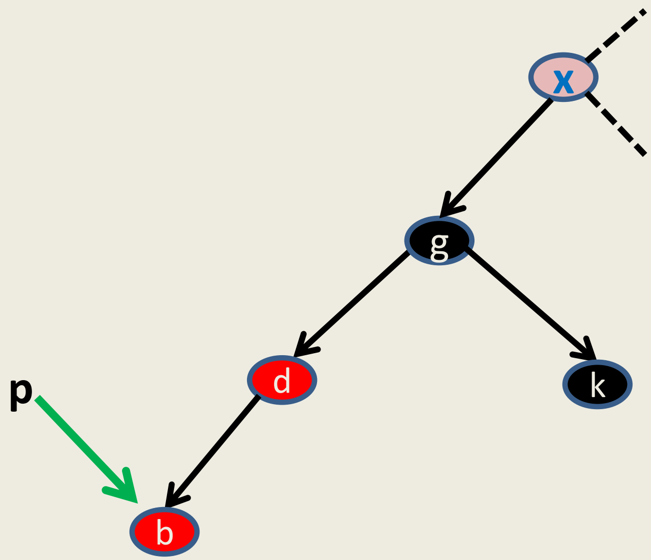
- *p* is a **red** colored node.
- ***parent**(p)* is also **red**.
- ***uncle**(p)* is **black**.

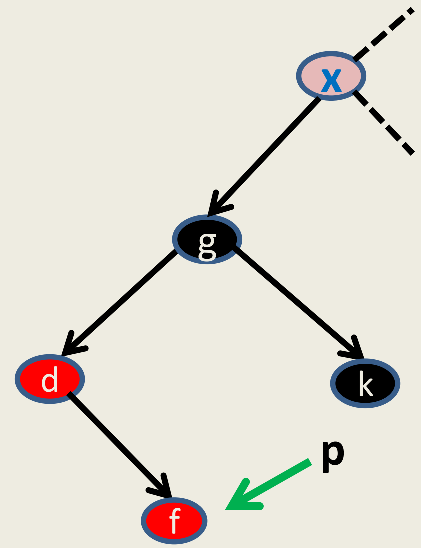
Without loss of generality assume: ***parent**(p)* is ***left child*** of ***grandparent**(p)*.

(The case when ***parent**(p)* is ***right child*** of ***grandparent**(p)* is handled similarly.)

# Handling the case 3

**two cases arise depending upon whether p is left/right child of its parent**



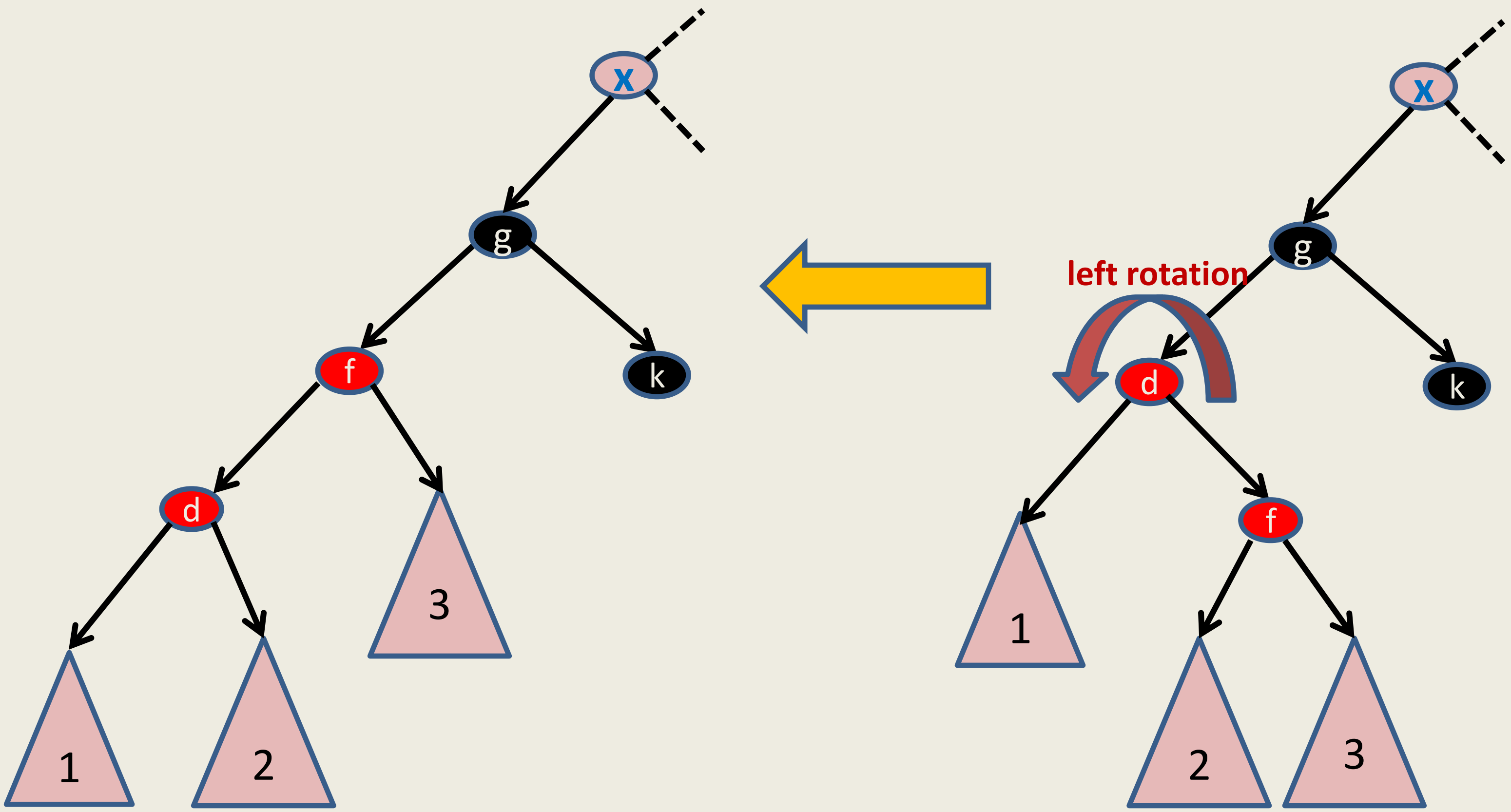

Can you transform
**Case 3.2** to
**Case 3.1** ?

**Case 3.1:**
p is **left child** of its **parent**

**Case 3.2:**
p is **right child** of its **parent**

# Handling the case 3

**two cases arise depending upon whether p is left/right child of its parent**

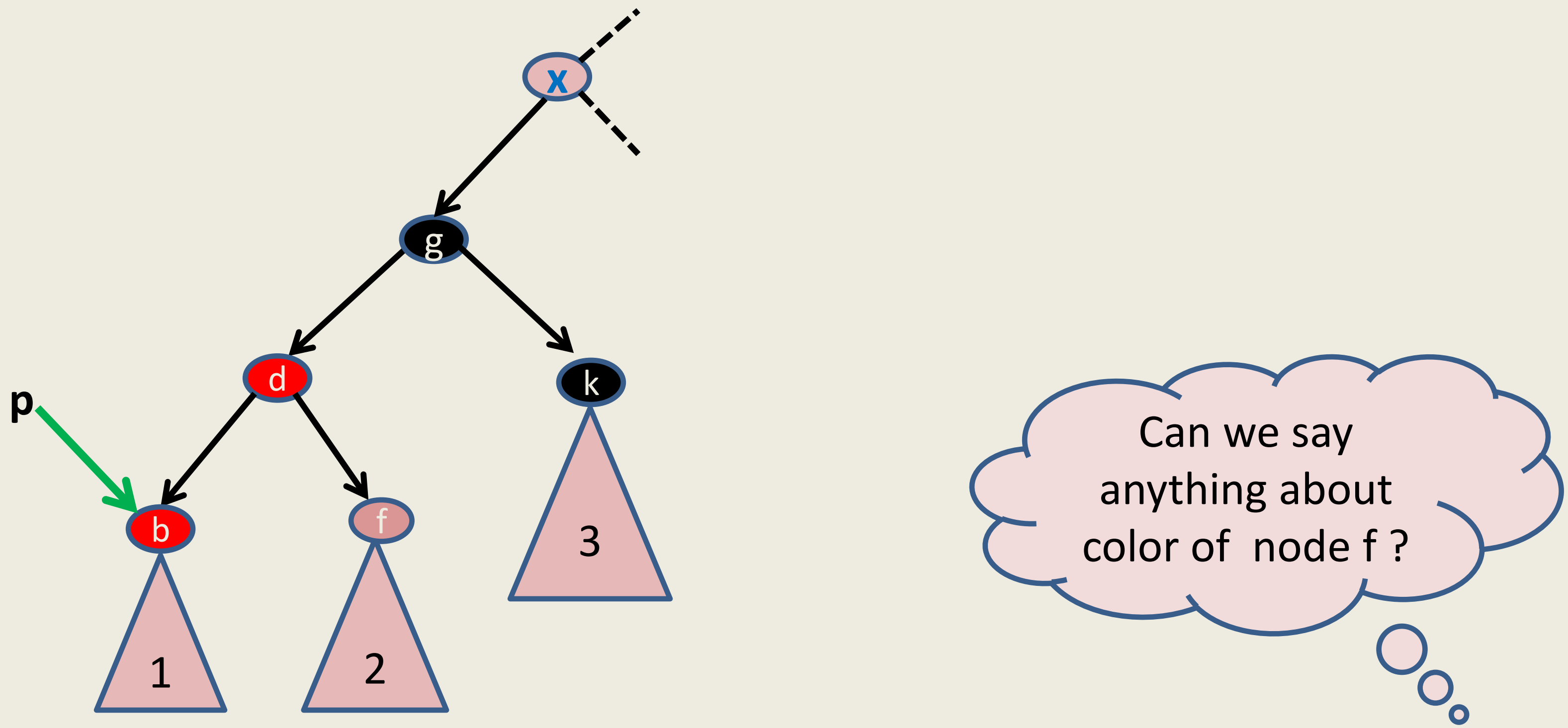



Vow!
This is exactly **Case 3.1**

**Case 3.2:**
p is **right child** of its **parent**

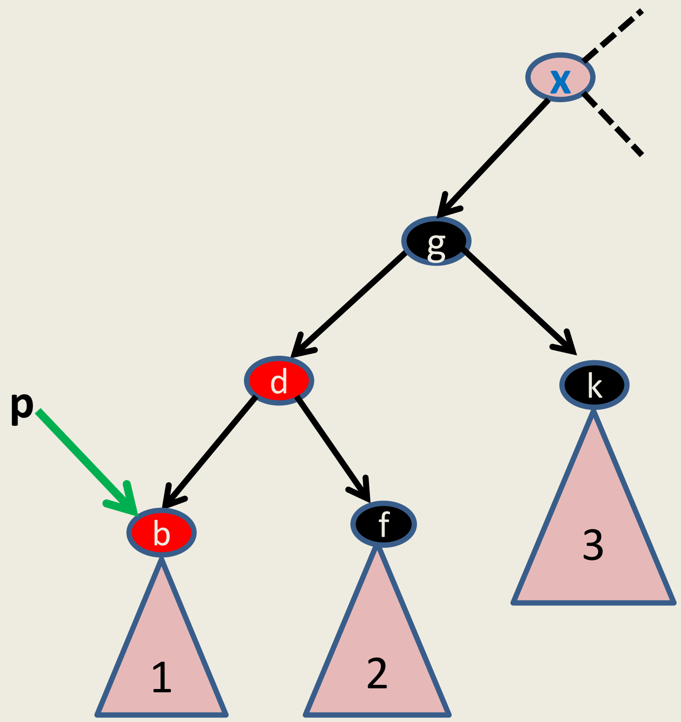
**We need to handle only case 3.1**

# Handling the case 3.1

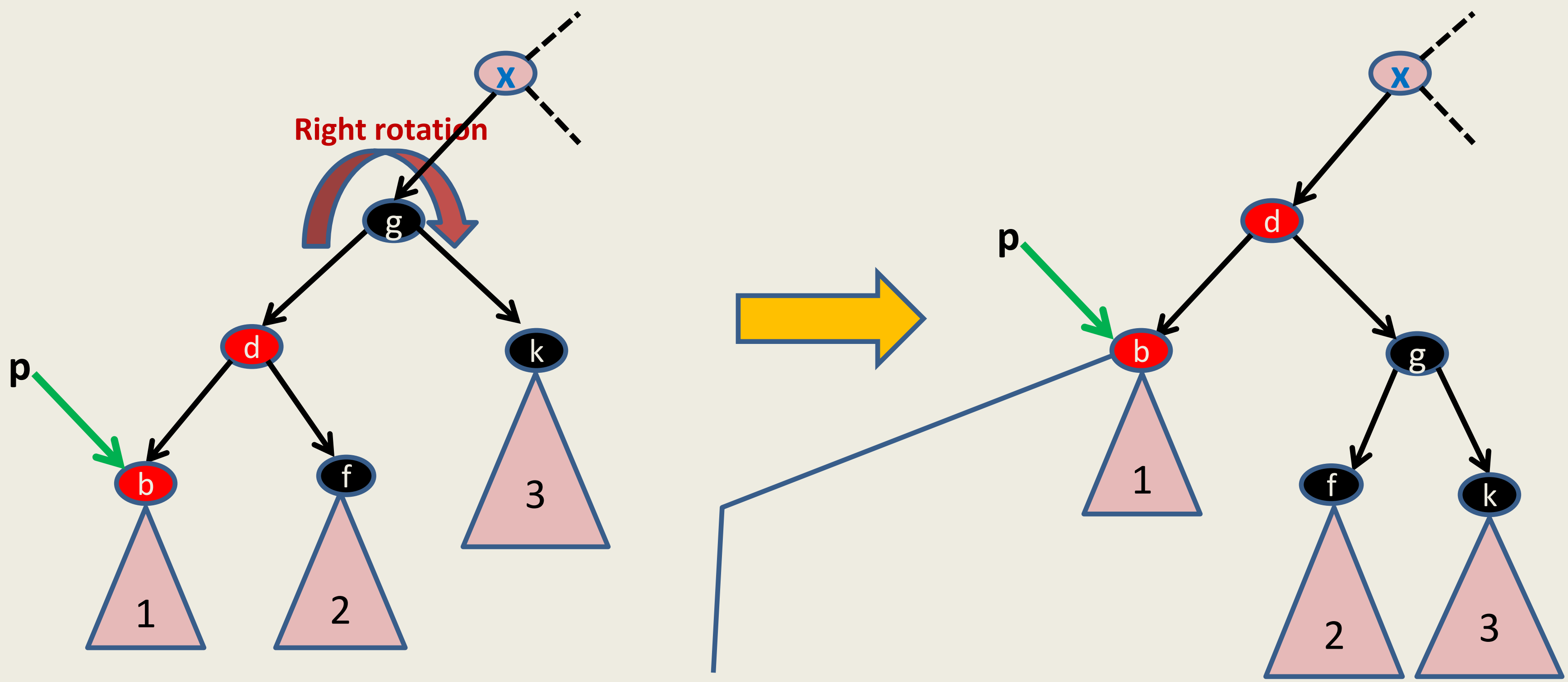



black

# Handling the case 3.1



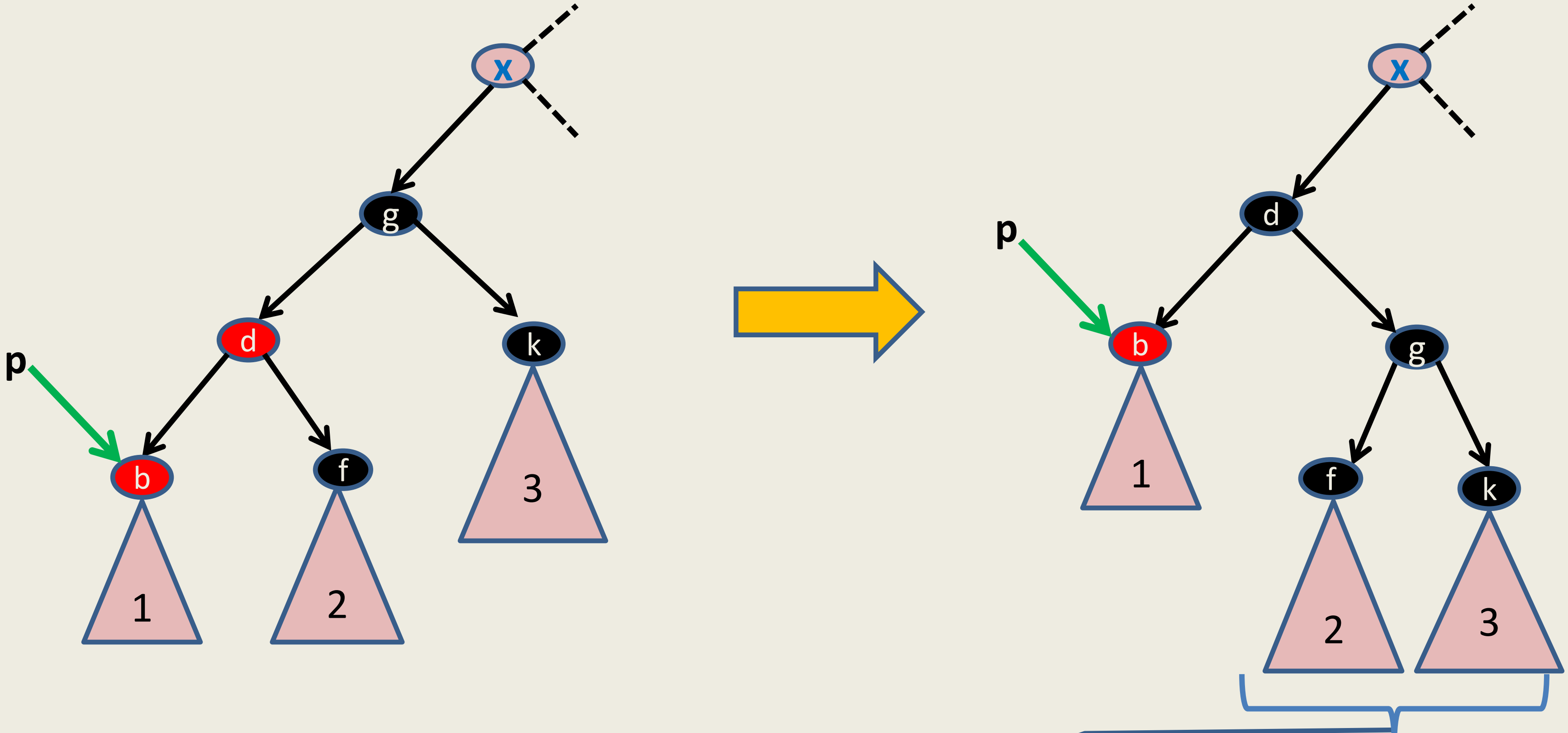
# Handling the case 3.1




Now every node in tree 1 has one less **black** node on the path to root ! We must restore it. Moreover, the color imbalance exists even now. What to do ?

Change color of node d to **black**

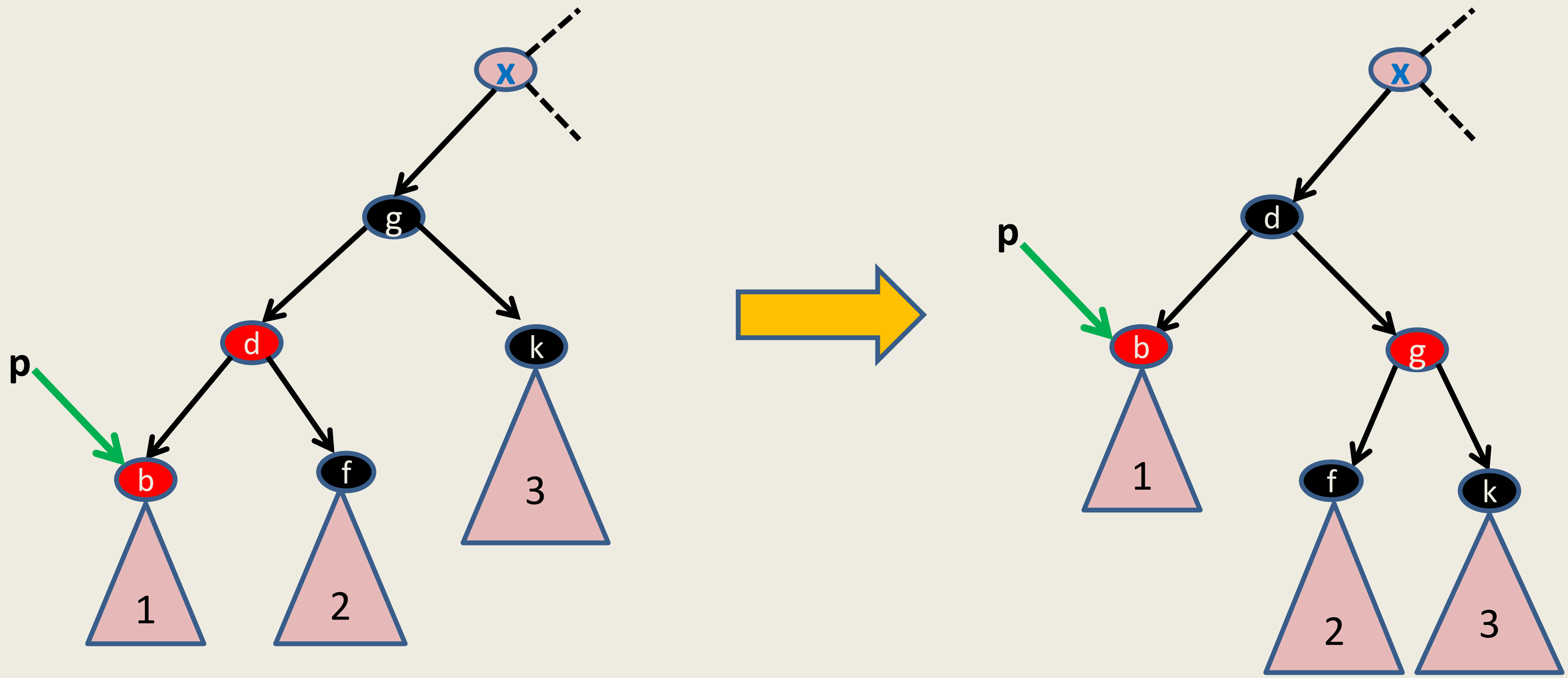
# Handling the case 3.1




The number of **black** nodes on the pat
restored for tree 1. Color imbalance is
But the number of **black** nodes on the pat
increased by one for trees 2 and 3. What to do now ?

Color node g **red**

# Handling the case 3.1




The black height is
restored for all trees.
This completes **Case 3.1**

**Theorem:**

We can maintain **red-black** trees under insertion of nodes in **O**(log $n$) time per <u>insert/search</u> operation where $n$ is the number of the nodes in the tree.

I hope you enjoyed the real fun in handling insertion in a **red** **black** tree.

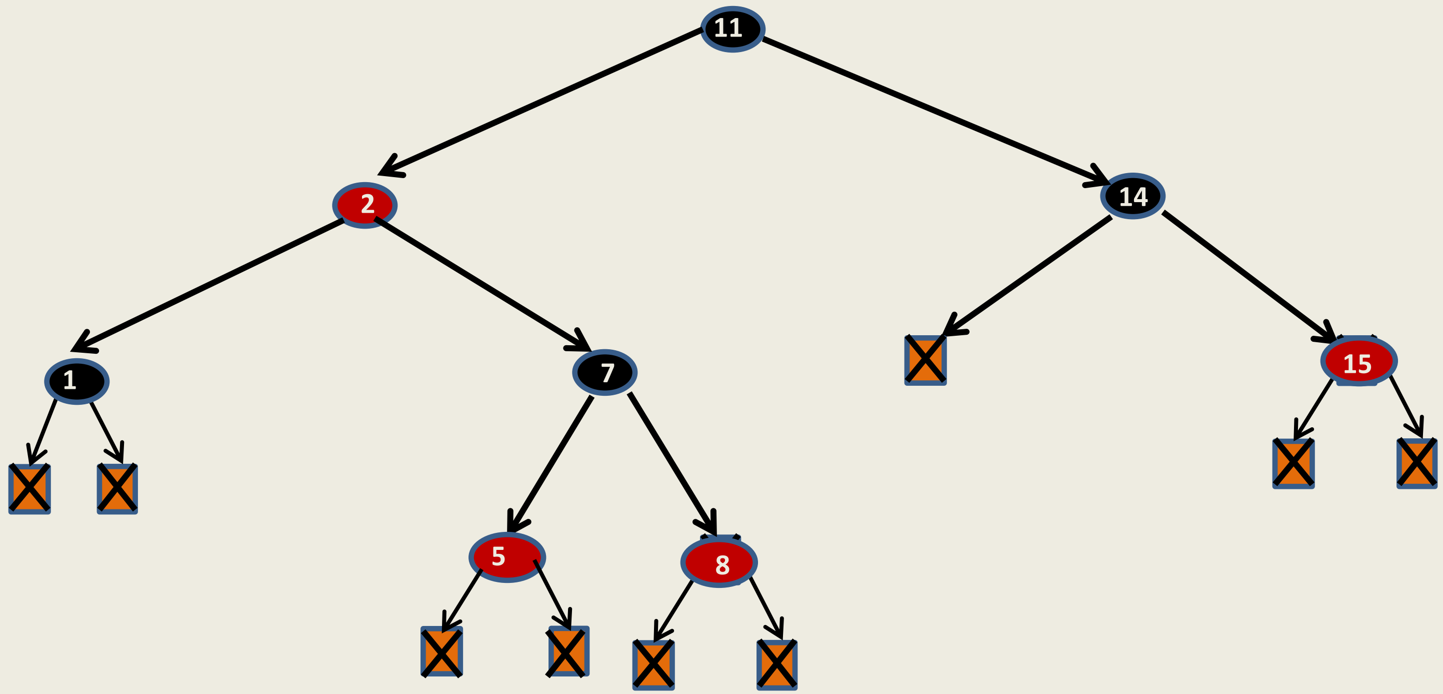
The following are the natural questions to ask.

- Why we are handling insertions in "this *particular* way" ?
- Are there *alternative and simpler* ways to handle insertions ?

You are encouraged to explore the answer to both these questions.
You are welcome to discuss them with me.

- Please solve the problem on the following slide.

# How to insert 4 ?



## How do will we handle deletion ?

This is going to be a bit more complex.

So please try on your own first before coming to the next class.

It will still involve playing with colors and rotations ☺

Do not miss the next class ☺