Data Structures and Algorithms

(CS210A)

Lecture 15:

- Algorithm paradigm of Divide and Conquer :

  Counting the number of Inversions
- Another sorting algorithm based on Divide and Conquer : Quick Sort

Divide and Conquer paradigm
An Overview

1. **Divide** the problem instance into two or more instances of the same problem
2. Solve each smaller instances **recursively** (base case suitably defined).
3. **Combine** the solutions of the smaller instances to get the solution of the original instance.

This is usually the main **nontrivial** step in the design of an algorithm using divide and conquer strategy

2 IMPORTANT LESSONS

THAT WE WILL LEARN TODAY…

1. Role of Data structures in algorithms
2. Learn from the past …

Role of Data Structures in designing efficient algorithms

Definition: A collection of data elements *arranged* and *connected* in a way which can facilitate efficient executions of a (possibly long) sequence of operations.

Parameters:

- Query/Update time
- Space
- Preprocessing time

Role of Data Structures in designing efficient algorithms

Definition: A collection of data elements *arranged* and *connected* in a way which can facilitate efficient executions of a (possibly long) sequence of operations.

Consider an Algorithm $\boldsymbol{A}$.
Suppose $\boldsymbol{A}$ performs many operations of **same type** on some data.

Improving time complexity of these operations ⟹ Improving the time complexity of $\boldsymbol{A}$.

So, it is worth designing a suitable **data structure**.

Counting Inversions in an array

Problem description

Definition (Inversion): Given an array **A** of size n, a pair (i,j), $0 \le i < j < n$ is called an inversion if **A**$[i] >$ **A**$[j]$.

Example:

	0	1	2	3	4	5	6	7
A	3	15	8	19	9	67	11	27

Inversions are :

(1,2), (1,4), (1,6),

(3,4),(3,6),

(5,6), (5,7)

AIM: An efficient algorithm to count the number of inversions in an array **A**.

Counting Inversions in an array

Problem familiarization

Trivial-algo(**A**[$\mathbf{0..}\boldsymbol{n}-\mathbf{1}$])

{ **count** ← 0;

  For($\boldsymbol{j}$=1 to $\boldsymbol{n}-\mathbf{1}$) **do**

  {   **For(** $\boldsymbol{i}$=0 to $\boldsymbol{j}-\mathbf{1}$ **)**

    {   **If** (**A**[$\boldsymbol{i}$]>**A**[$\boldsymbol{j}$]) **count** ← **count** + 1;

    }

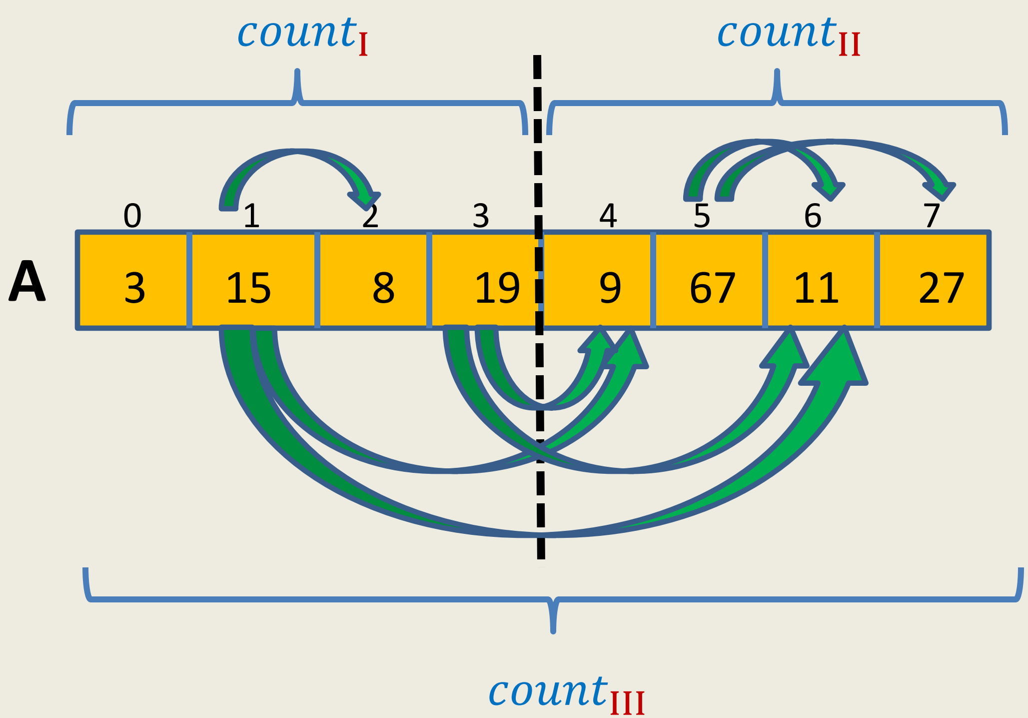
  } return **count;**

}

Time complexity: **O**($\boldsymbol{n}^{\mathbf{2}}$)

Let us try to design a
Divide and Conquer based algorithm

How do we approach using divide & conquer



Counting Inversions
Divide and Conquer based algorithm

CountInversion(**A**,$\boldsymbol{i}, \boldsymbol{k}$)   // Counting no. of inversions in **A**[$\boldsymbol{i}..\boldsymbol{k}$]

If ($\boldsymbol{i} = \boldsymbol{k}$) return 0;

Else{ $\mathbf{mid} \leftarrow (\boldsymbol{i} + \boldsymbol{k})/2$;

$count_{\mathrm{I}} \leftarrow$ **CountInversion**(**A**,$\boldsymbol{i}, \mathbf{mid}$);

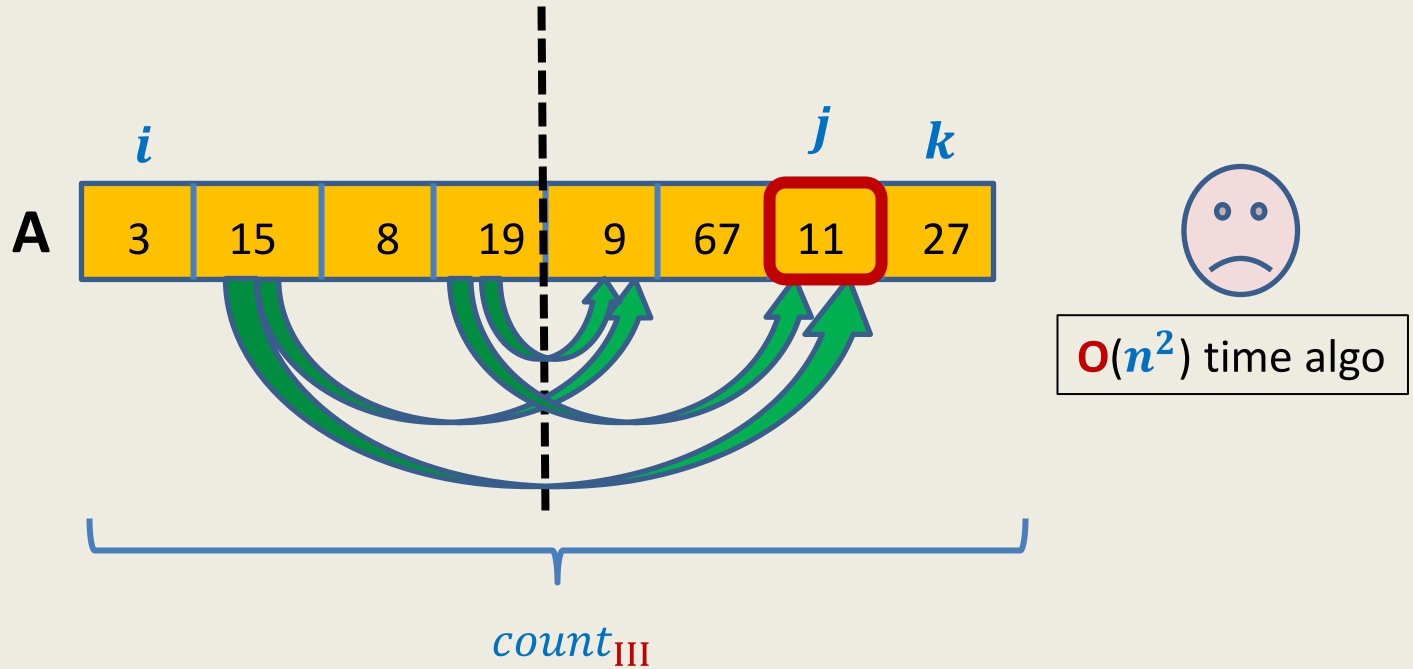
$count_{\mathrm{II}} \leftarrow$ **CountInversion**(**A**,$\mathbf{mid} + \mathbf{1}, \boldsymbol{k}$);

…. *Code for* $count_{\mathrm{III}}$ ….

return $count_{\mathrm{I}}$ + $count_{\mathrm{II}}$ + $count_{\mathrm{III}}$;

}

How to efficiently compute $count_{\mathrm{III}}$ (Inversions of type III) ?



Aim: For each $\mathbf{mid} < j \leq k$, count the elements in **A**[i..$\mathbf{mid}$] that are **greater** than **A**[j].

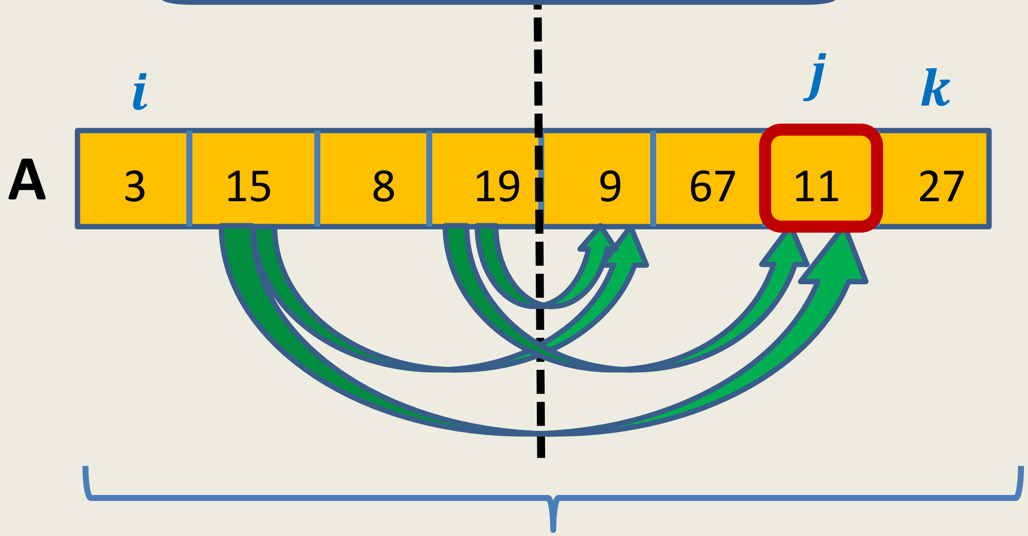
Trivial way: **O**(**size of the subarray A**[i..$\mathbf{mid}$]) time for a given j.

→**O**(n) time for a given j in the first call of the algorithm.

→**O**(n^2) time for computing $count_{\mathrm{III}}$ since there are $n/2$ possible values of j.

How to efficiently compute $count_{\mathrm{III}}$ (Inversions of type III) ?

Let us state clearly what we want to achieve …



$count_{\mathrm{III}}$

For each $\boldsymbol{j} \in [\boldsymbol{mid+1}, \boldsymbol{k}]$

count the elements in **A**[$\boldsymbol{i}$..$\mathbf{mid}$] that are **greater** than **A**[$\boldsymbol{j}$].

Time to apply Lesson 1

What should be the **data structure** ?

Sorted subarray **A**[$\boldsymbol{i}$..$\mathbf{mid}$].

Counting Inversions
First algorithm based on divide & conquer

CountInversion(**A**,i, k)

If (i=k) return 0;

Else{ $\mathbf{mid}$ ← ($i + k$)/2;

$count_{\mathrm{I}}$ ← **CountInversion**(**A**,i, $\mathbf{mid}$);

$count_{\mathrm{II}}$ ← **CountInversion**(**A**,$\mathbf{mid} + 1$, k);

2 T($n/2$)

Sort(**A**,i, $\mathbf{mid}$);

For each $\mathbf{mid} < j \leq k$

do **binary search** for **A**[j] in **A**[i..$\mathbf{mid}$] to compute the *number* of elements greater than **A**[j].

Add this *number* to $count_{\mathrm{III}}$;

c n **log** n

return $count_{\mathrm{I}}$ + $count_{\mathrm{II}}$ + $count_{\mathrm{III}}$;

}

Counting Inversions

First algorithm based on divide & conquer

Time complexity analysis:

If $n = 1$,

$\quad$ $T(n) = c$ for some constant c

If $n > 1$,

$\quad$ $T(n) = c\, n \log n + 2\, T(n/2)$

$\quad\quad = c\, n \log n + c\, n\, ((\log n)-1) + 2^2\, T(n/2^2)$

$\quad\quad = c\, n \log n + c\, n\, ((\log n)-1) + c\, n\, ((\log n)-2) + 2^3\, T(n/2^3)$

$\quad\quad = O(n \log^2 n)$

Can we improve it further ?

Counting Inversions
First algorithm based on divide & conquer

CountInversion(**A**,i, k)

If (i=k) return 0;

Else{ $\mathbf{mid} \leftarrow (i+k)/2$;

$count_{\mathrm{I}}$ ← **CountInversion**(**A**,i, $\mathbf{mid}$);

$count_{\mathrm{II}}$ ← **CountInversion**(**A**,$\mathbf{mid}+1, k$);

2 **T**($n/2$)

Sort(**A**,i, $\mathbf{mid}$);

For each $\mathbf{mid} < j \le k$

do **binary search** for **A**[j] in **A**[i..$\mathbf{mid}$] to compute the *number* of elements greater than **A**[j].

Add this *number* to $count_{\mathrm{III}}$;

c n **log** n

return $count_{\mathrm{I}} + count_{\mathrm{II}} + count_{\mathrm{III}}$;

}

Sequence of observations
To achieve better running time

- The extra **log** n factor arises because for the "**combine**" step, we are spending **O**(n **log** n) time instead of **O**(n).

- The reason for **O**(n **log** n) time for the "**combine**" step:
  - **Sorting A**[0.. n/2] takes **O**(n **log** n) time.
  - Doing **Binary Search** for n/**2** elements from **A**[n**/2… n -1**]

- Each of the above tasks have optimal running time.

- So the only way to improve the running time of "**combine**" step is some new idea

Revisiting MergeSort algorithm

MSort(**A**,$\boldsymbol{i}$, $\boldsymbol{k}$)// Sorting **A**[$\boldsymbol{i}..\boldsymbol{k}$]

{ **If** ($\boldsymbol{i} < \boldsymbol{k}$)

{ $\mathbf{mid}$ ← ($\boldsymbol{i} + \boldsymbol{k}$)/2;

MSort(**A**,$\boldsymbol{i}$, $\mathbf{mid}$);

MSort(**A**,$\boldsymbol{mid} + \mathbf{1}$, $\boldsymbol{k}$);

Create a temporary array **C**[$\mathbf{0}..\boldsymbol{k} - \boldsymbol{i}$]

Merge(**A**,$\boldsymbol{i}$, $\boldsymbol{mid}$, $\boldsymbol{k}$, **C**);

Copy **C**[$\mathbf{0}..\boldsymbol{k} - \boldsymbol{i}$] to **A**[$\boldsymbol{i}..\boldsymbol{k}$]

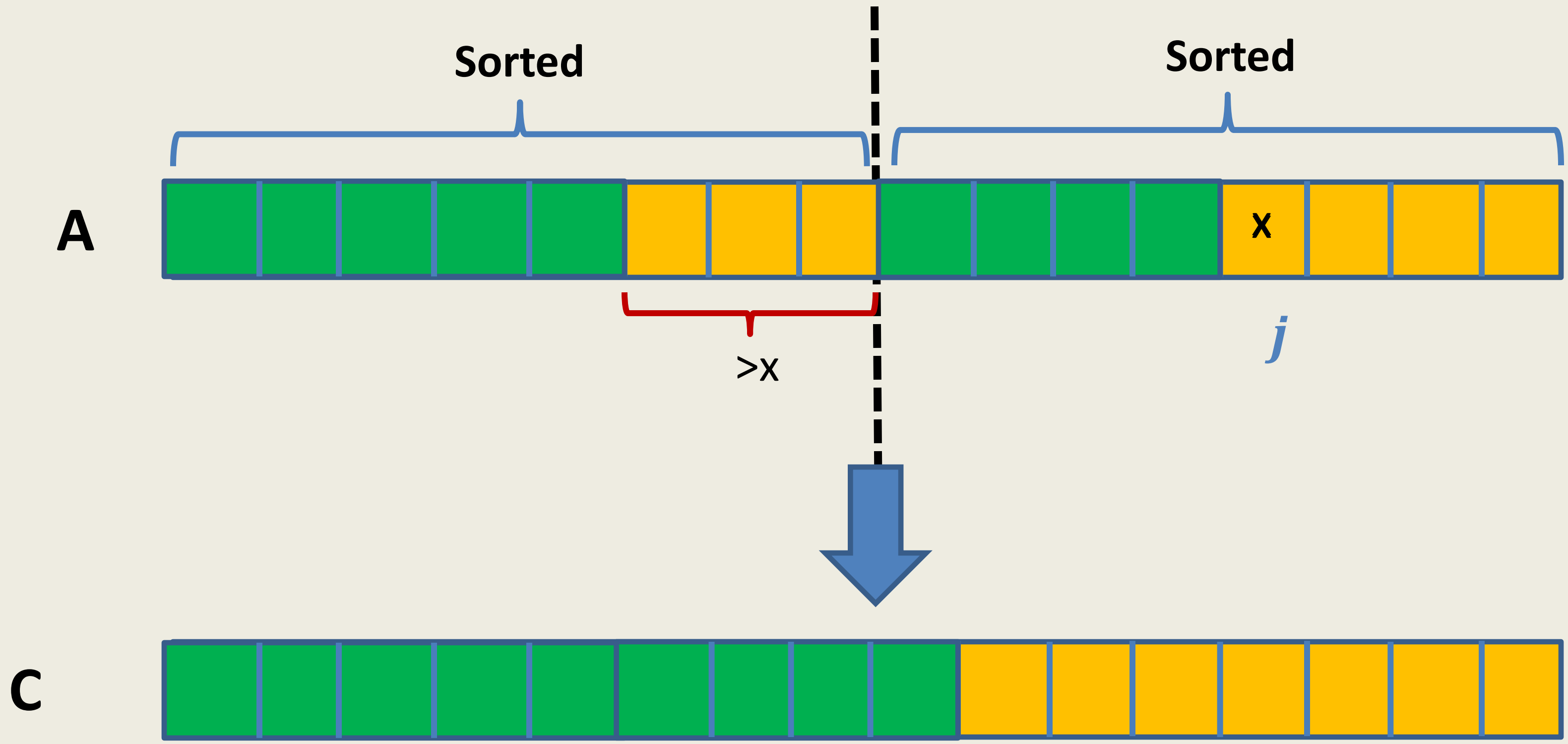
}

}

We shall carefully look at the **Merge**() procedure to find an efficient way to count the number of elements from **A**[$\boldsymbol{i}..\mathbf{mid}$] which are smaller than **A**[$\boldsymbol{j}$] for any given $\mathbf{mid} < \boldsymbol{j} \leq \boldsymbol{k}$

Relook

Merging A[$i..\mathbf{mid}$] and A[$\mathbf{mid}+\mathbf{1}..k$]



Pesudo-code for Merging two sorted arrays

Merge(**A**,$\boldsymbol{i}$, $\mathbf{mid}$, $\boldsymbol{k}$,**C**)

$\boldsymbol{p} \leftarrow \boldsymbol{i}$; $\boldsymbol{j} \leftarrow \mathbf{mid} + \mathbf{1}$; $\boldsymbol{r} \leftarrow 0$;

While($\boldsymbol{p} \le \mathbf{mid}$ and $\boldsymbol{j} \le \boldsymbol{k}$)

{ **If**(**A**[$\boldsymbol{p}$]< **A**[$\boldsymbol{j}$]) **{** **C**[$\boldsymbol{r}$] $\leftarrow$ **A**[$\boldsymbol{p}$]; $\boldsymbol{r}$++; $\boldsymbol{p}$++ **}**

Else **{** **C**[$\boldsymbol{r}$] $\leftarrow$ **A**[$\boldsymbol{j}$]; $\boldsymbol{r}$++; $\boldsymbol{j}$++ **}**

}

While($\boldsymbol{p} \le \mathbf{mid}$) **{** **C**[$\boldsymbol{k}$] $\leftarrow$ **A**[$\boldsymbol{i}$]; $\boldsymbol{k}$++; $\boldsymbol{i}$++ **}**

While($\boldsymbol{j} \le \boldsymbol{k}$) **{** **C**[$\boldsymbol{k}$] $\leftarrow$ **A**[$\boldsymbol{j}$]; $\boldsymbol{k}$++; $\boldsymbol{j}$++ **}**

return **C ;**

We shall make **just a slight change** in the above pseudo-code to achieve our main objective of computing $count_{\mathrm{III}}$. If you understood the discussion of the previous slide, can you guess it now ?

Pesudo-code for
Merging and counting inversions

```
Merge_and_CountInversion(A, i, mid, k, C)
  p ← i;  j ← mid + 1; r ← 0;
  count_III ← 0;
  While(p ≤ mid  and j ≤ k)
  {      If(A[p]< A[j]) {    C[r] ← A[p]; r++;  p++  }
         Else            {    C[r] ← A[j]; r++;  j++
                              count_III ← count_III +  (mid − p + 1);
                         }
  }
  While(p ≤ mid)    { C[k] ← A[i]; k++;  i++   }
  While(j ≤ k)      { C[k] ← A[j]; k++;  j++   }
  return count_III;
```

Nothing extra is needed here.

Counting Inversions
Final algorithm based on divide & conquer

Sort_and_CountInversion(A, i, k)

{ If ($i = k$) return 0;

else

{ $\mathbf{mid} \leftarrow (i + k)/2$;

$count_{\mathrm{I}} \leftarrow$ Sort_and_CountInversion (A, i, $\mathbf{mid}$);

$count_{\mathrm{II}} \leftarrow$ Sort_and_CountInversion (A, $\mathbf{mid} + 1$, k);

$\}$ 2 T($n/2$)

Create a temporary array C[$0..k - i$]

$count_{\mathrm{III}} \leftarrow$ Merge_and_CountInversion(A, i, $\mathbf{mid}$, k, C);

Copy C[$0..k - i$] to A[$i..k$];

return $count_{\mathrm{I}}$ + $count_{\mathrm{II}}$ + $count_{\mathrm{III}}$;

$\}$ O(n)

}

}

Counting Inversions
Final algorithm based on divide & conquer

Time complexity analysis:

If n = 1,

T(n) = c for some constant c

If n > 1,

T(n) = c n + **2 T**(n/**2**)

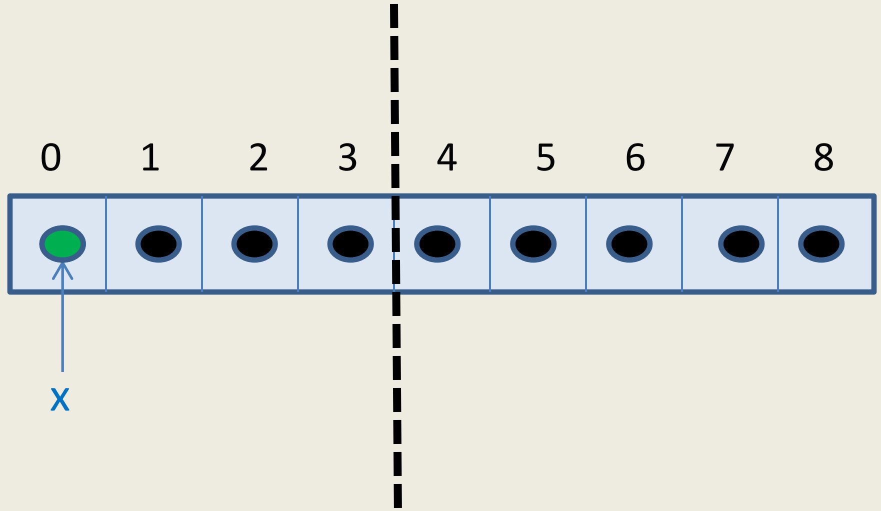
= **O**(n **log** n)

Theorem: There is **a divide and conquer** based algorithm for computing the number of inversions in an array of size n. The running time of the algorithm is **O**(n **log** n).

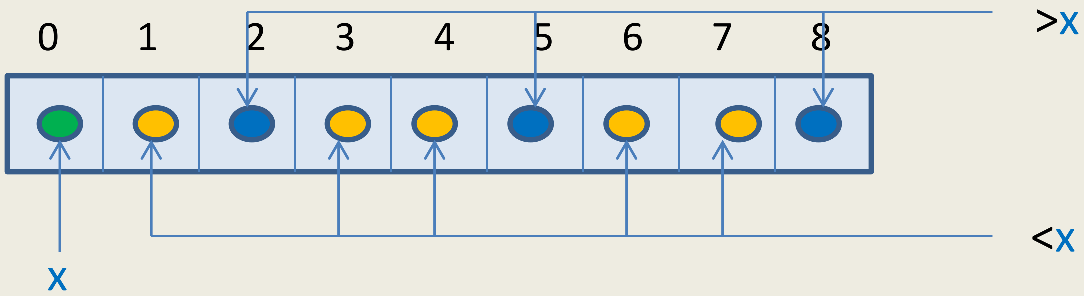
Another sorting algorithm based on divide and conquer

QuickSort

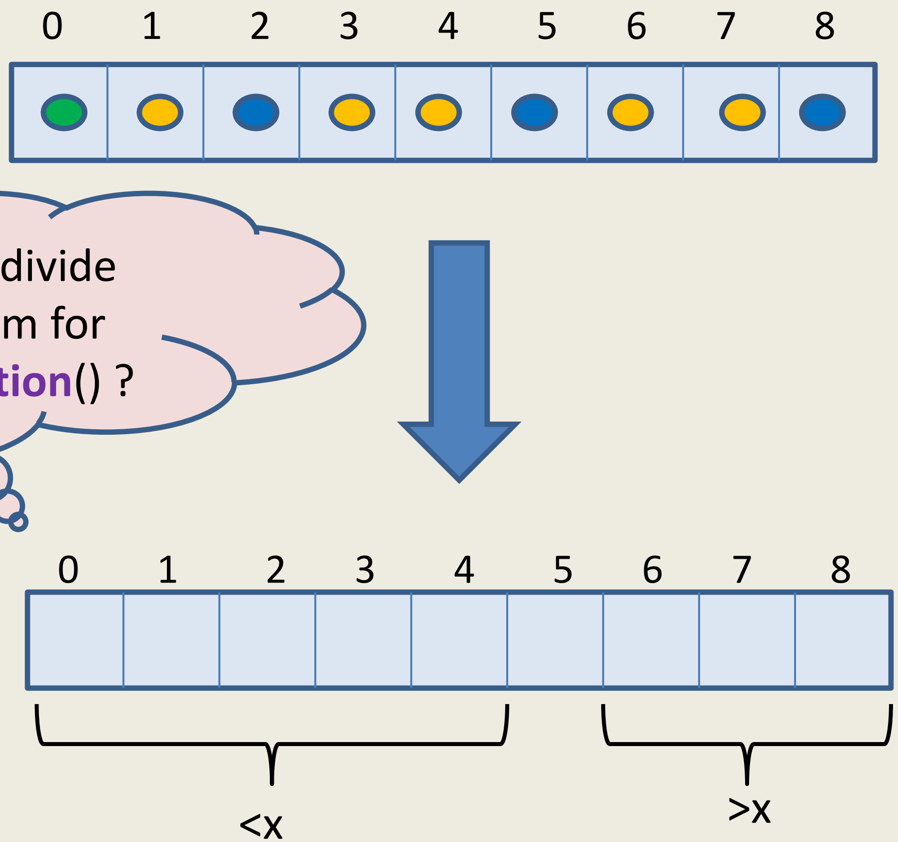
Is there any alternate way to divide ?



In MergeSort, we divide the input instance in an obvious manner.


0
1
2
3
4
5
6
7
8
>x
<x
x

Can you now guess a divide and conquer algorithm for sorting based on **Partition**() ?



This procedure is called **Partition**.

It **rearranges** the elements so that all elements less than x appear to the left of x and all elements greater than x appear to the right of x.

Pseudocode for QuickSort(S)

QuickSort(S)
{ If (|S|>1)
Pick and remove an element x from S;
($S_{<x}$, $S_{>x}$) ← Partition(S,x);
return(Concatenate(QuickSort($S_{<x}$), x, QuickSort($S_{>x}$))
}

Pseudocode for QuickSort(S)

When the input S is stored in an array

QuickSort(A,l, r)

{ If (l < r)

i ← Partition(A,l,r); // i is index where element $A[l]$ is finally placed

QuickSort(A,l, $i-1$);

QuickSort(A,$i+1$, r)

}

View this algorithm from various perspectives. For almost all practical purposes, this is the most efficient algorithm for sorting. It outperforms **MergeSort** by a significant factor.

QuickSort

Homework:

- The running time of Quick Sort depends upon
  the element we choose for partition in each recursive call.
- What can be the worst case running time of Quick Sort ?
- What can be the best case running time of Quick Sort ?

- Give an implementation of **Partition** that takes **O**($r - l$) time and using **O**(1) extra space only. (I gave this homework last week as well)

Sometime later in the course, we shall revisit ***QuickSort*** *and analyze it theoretically (average time complexity) and experimentally.*

The outcome will be surprising and counterintuitive. ☺