Data Structures and Algorithms

(CS210A)

Semester I – **2014-15**

Lecture 35

- **A new algorithm design paradigm:** Greedy strategy

part II

Continuing Problem from last class

JOB Scheduling

Largest subset of non-overlapping job

A job scheduling problem

Formal Description

INPUT:

- A set J of n jobs $\{j_1, j_2, \ldots, j_n\}$
- job j_i is specified by two real numbers

  s(i): start time of job j_i

  f(i): finish time of job j_i
- A single server

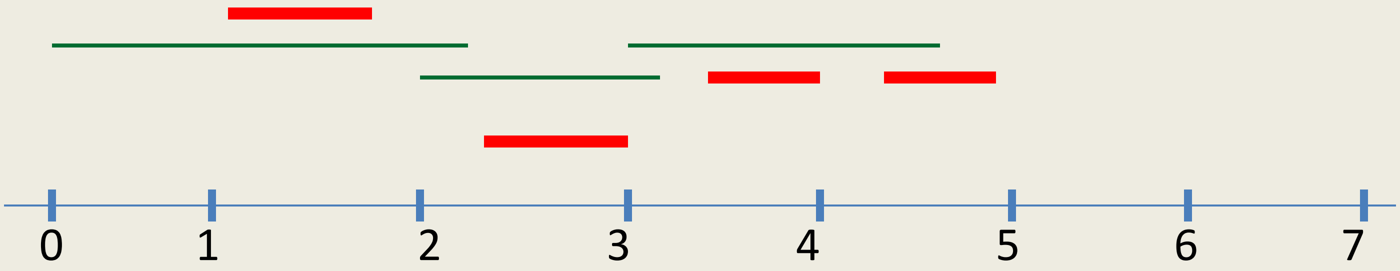
Constraints:

- Server can execute <u>at most one job </u>at any moment of time and a job.
- **Job** j_i, if scheduled, has to be scheduled <u>during[**s**(i), **f**(i)] only</u>.

Aim: To select the **largest** subset of **<u>non-overlapping</u>** jobs which can be executed by the server.

Designing algorithm for the problem

Strategy 4: **Select the job with earliest finish time**



Intuition:

Selecting such a job will **free** the server **earliest**

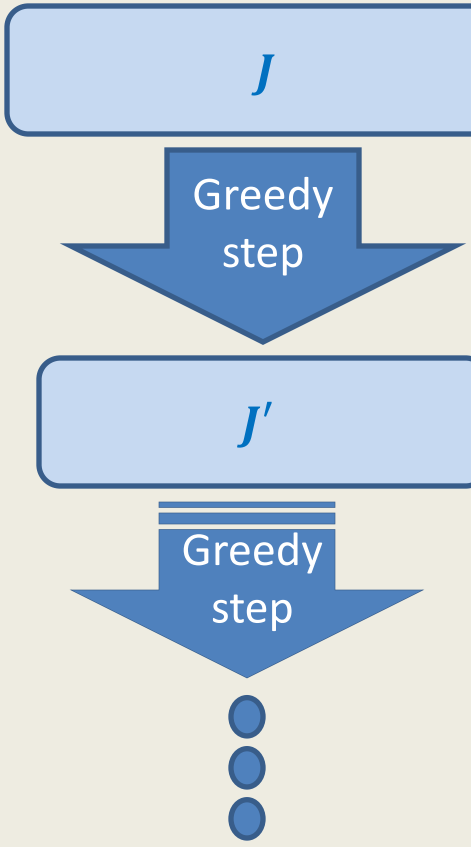
➔ hence more no. of jobs might get scheduled.

Algorithm "earliest finish time"

Proof of correctness ?

Let $x \in J$ be the job with earliest finish time.

Let $J' = J \backslash$ **Overlap**(x)



Algorithm **(Input :** set J of n jobs.)

1. Define $A \leftarrow \emptyset$;
2. **While** J <>$\emptyset$ **do**

   { Let $x \in J$ has earliest finish time;

   $A \leftarrow A \cup \{x\}$;

   $J \leftarrow J \backslash$**Overlap**$(x)$;

   }
3. Return A;

Lemma1 (last class): There exists <u>an</u> optimal solution for J in which x is present.

Algorithm "earliest finish time"

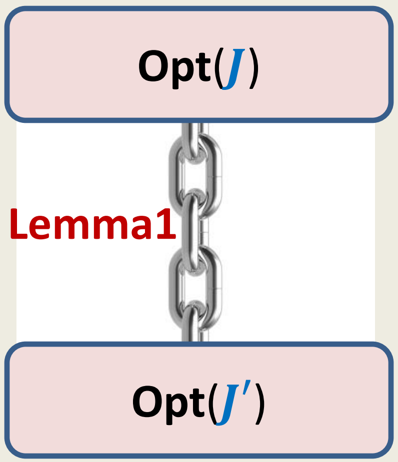
How to prove it ?

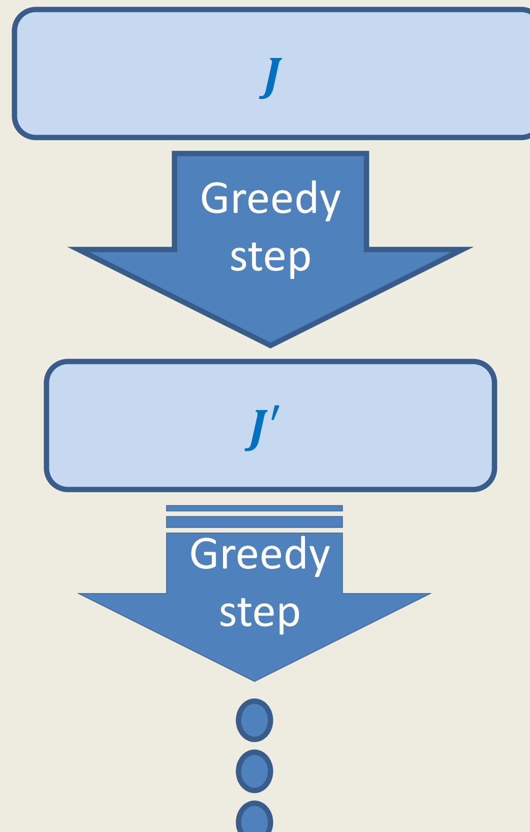
Proof of correctness ?

Let $x \in J$ be the job with earliest finish time.

Let $J' = J \backslash$**Overlap**(x)

Opt(J)= Opt(J') + 1





Notation:

Opt(J): the size of an optimal solution for J.

Theorem: Opt(J) = Opt(J') + 1.

- Proof has two parts

  Opt(J) ≥ **Opt**(J') + 1
  Opt(J') ≥ **Opt**(J) − 1

  Try to give a physical interpretation to these inequalities.

- Proof for each part is a proof by **construction**

Algorithm "earliest finish time"

Proving $\textbf{Opt}(J) \geq \textbf{Opt}(J') + 1$.

Observation: start time of every job in J' is greater than finish time of x.

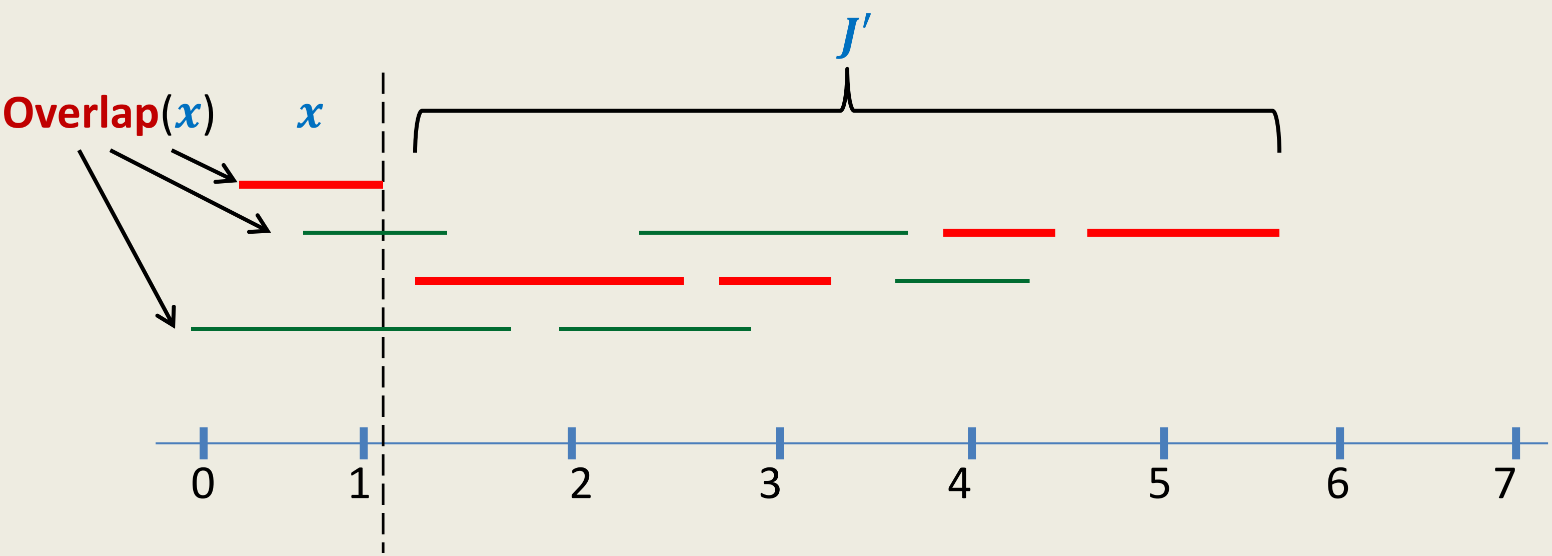
Let O' be any optimal solution for J'.

None of the jobs in O' overlaps with x.

Hence $O' \cup \{x\}$ is a

Therefore $\textbf{Opt}(J) \geq |O'| + 1 = \textbf{Opt}(J') + 1$.

From an **optimal solution** of J'
can you derive a **solution** for J with one extra job?



Algorithm "earliest finish time"

Proving $\textbf{Opt}(J') \geq \textbf{Opt}(J) - 1$.

Lemma1 (last class): There exists an optimal solution for J in which x is present.

Let O be an optimal solution for J containing x.

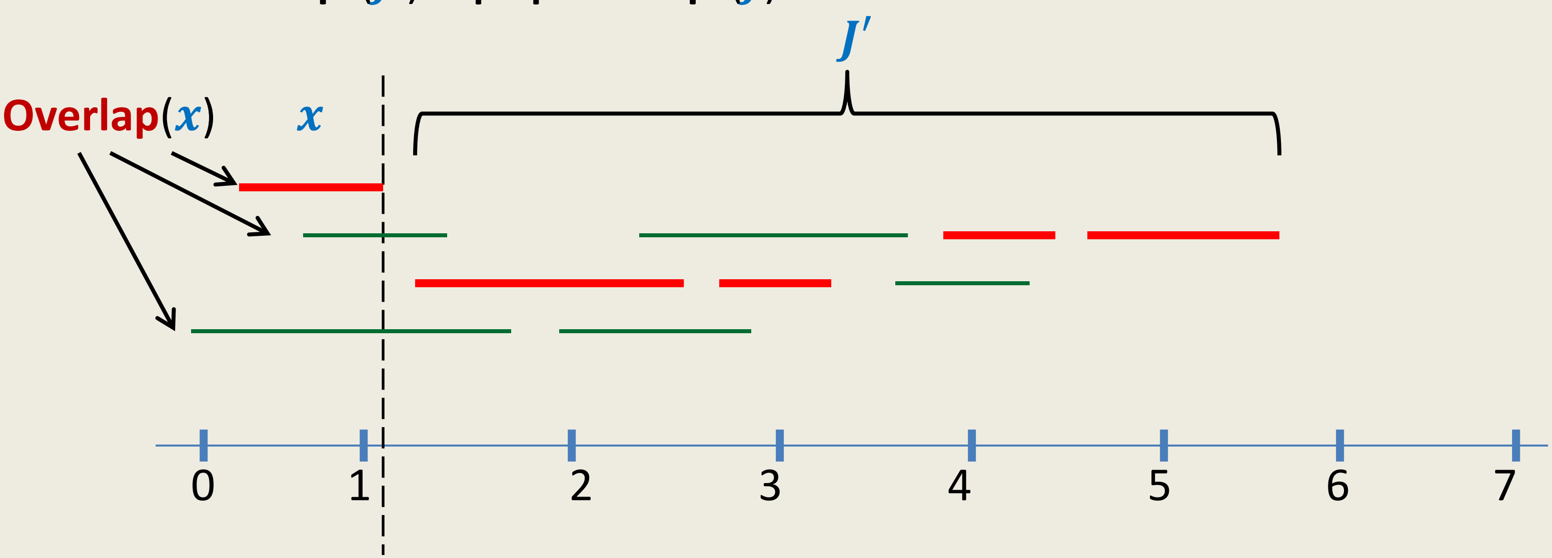
None of the jobs in

➔ Every job from

Hence $O \backslash \{x\}$ is a subset of non-overlapping jobs from J'.

Therefore $\textbf{Opt}(J') \geq |O| - 1 = \textbf{Opt}(J) - 1$.

From an **optimal solution** of J can you derive a **solution** for J' with one job less?



Theorem:

Given any set J of n jobs, the algorithm based on "**earliest finish time**" approach computes the largest subset of non-overlapping job.

O(n log n) implementation of the Algorithm

Algorithm **(Input :** set J of n jobs.)

1. Define $A \leftarrow \emptyset$;
2. **While J <>$\emptyset$ do**

   { Let $x \in J$ have earliest finish time;

   $A \leftarrow A \cup \{x\}$;

   $J \leftarrow J \backslash$**Overlap**($x$);

   }
3. Return A;

Maintain a **binary min-heap** for J based on **finish time** as the key.

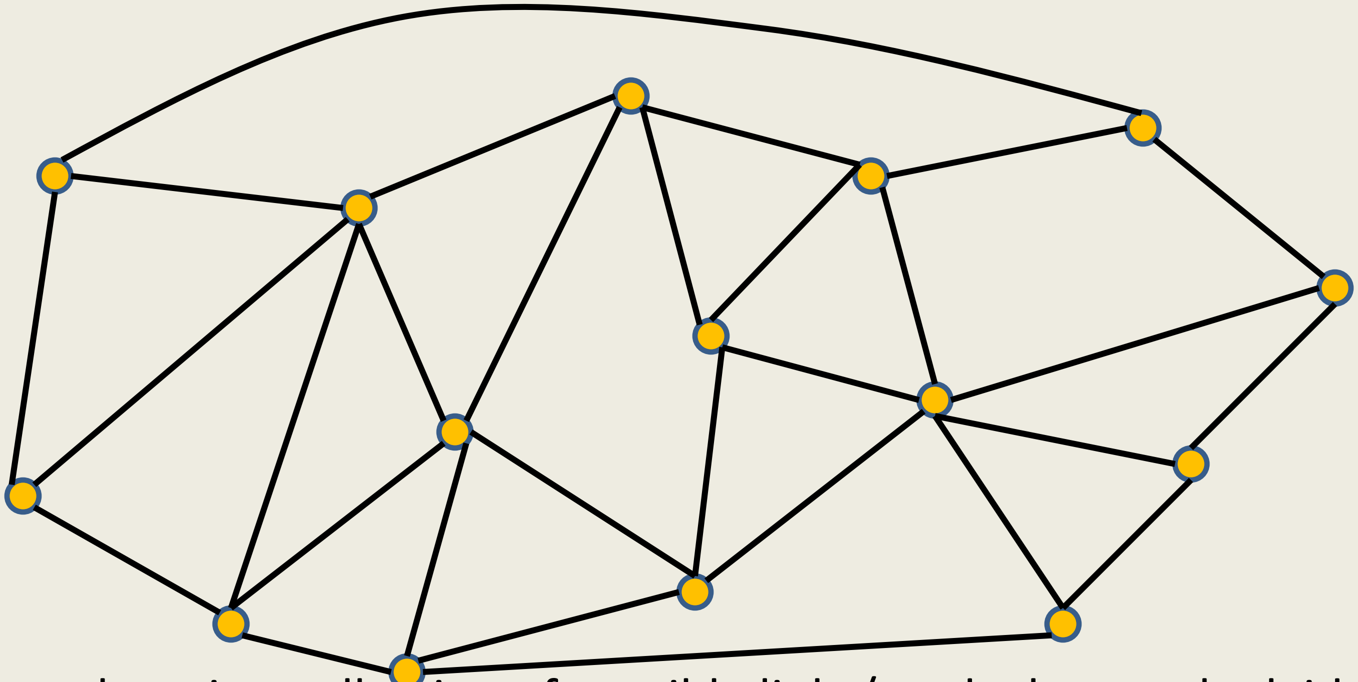
Sort J in increasing order of **start time**.

➔ **O(n^2) time complexity is obvious**

Problem 2

First we shall give motivation.

Motivation:
A road or telecommunication network



Suppose there is a collection of possible links/roads that can be laid.

But laying down each possible link/road is costly.

Aim: To lay down **least number** of links/roads to ensure **connectivity** between each pair of nodes/cities.

Motivation

Formal description of the problem

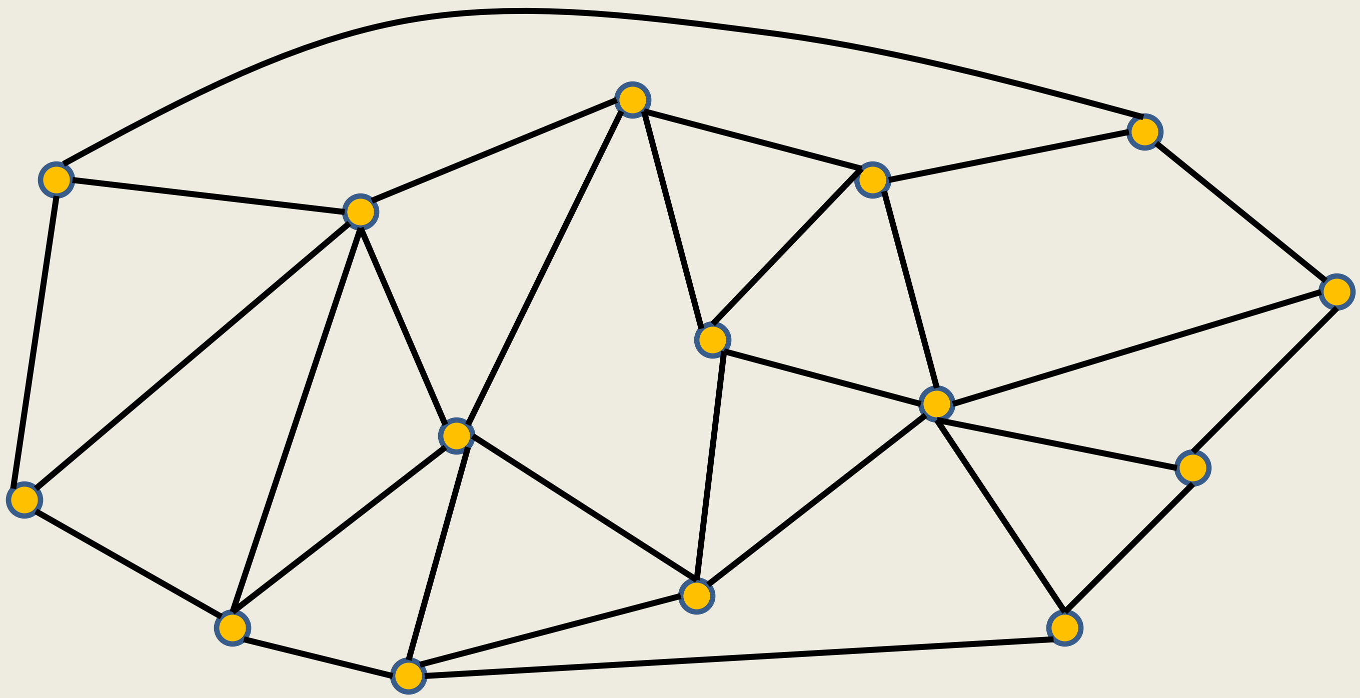
Input: an undirected graph **G**=(**V**,**E**).

Aim: compute a **subgraph** (**V**,**E'**), **E'** ⊆ **E** such that

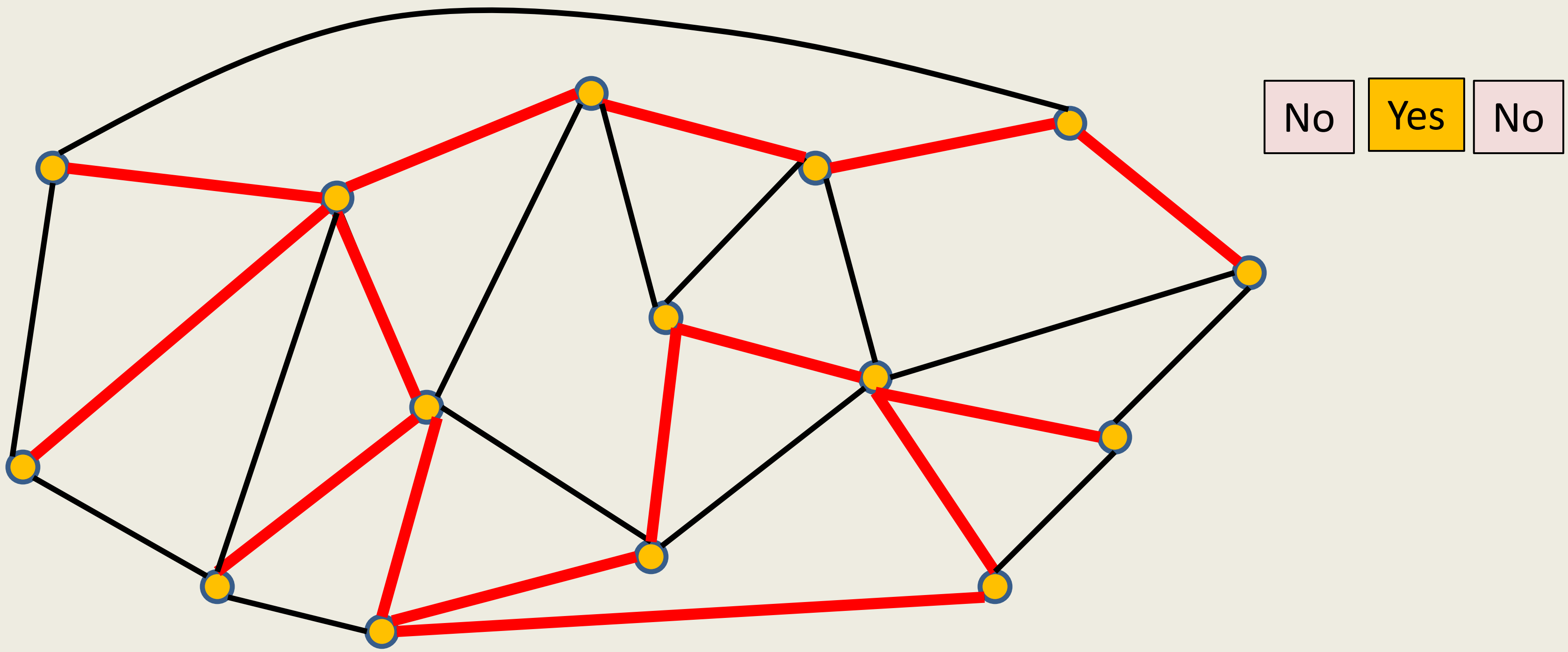
- **Connectivity** among all **V** is guaranteed in the **subgraph**.
- **|E'|** is **minimum.**

How will such a subgraph look like ?

A road or telecommunication network



A road or telecommunication network



Is this subgraph meeting our requirement ?

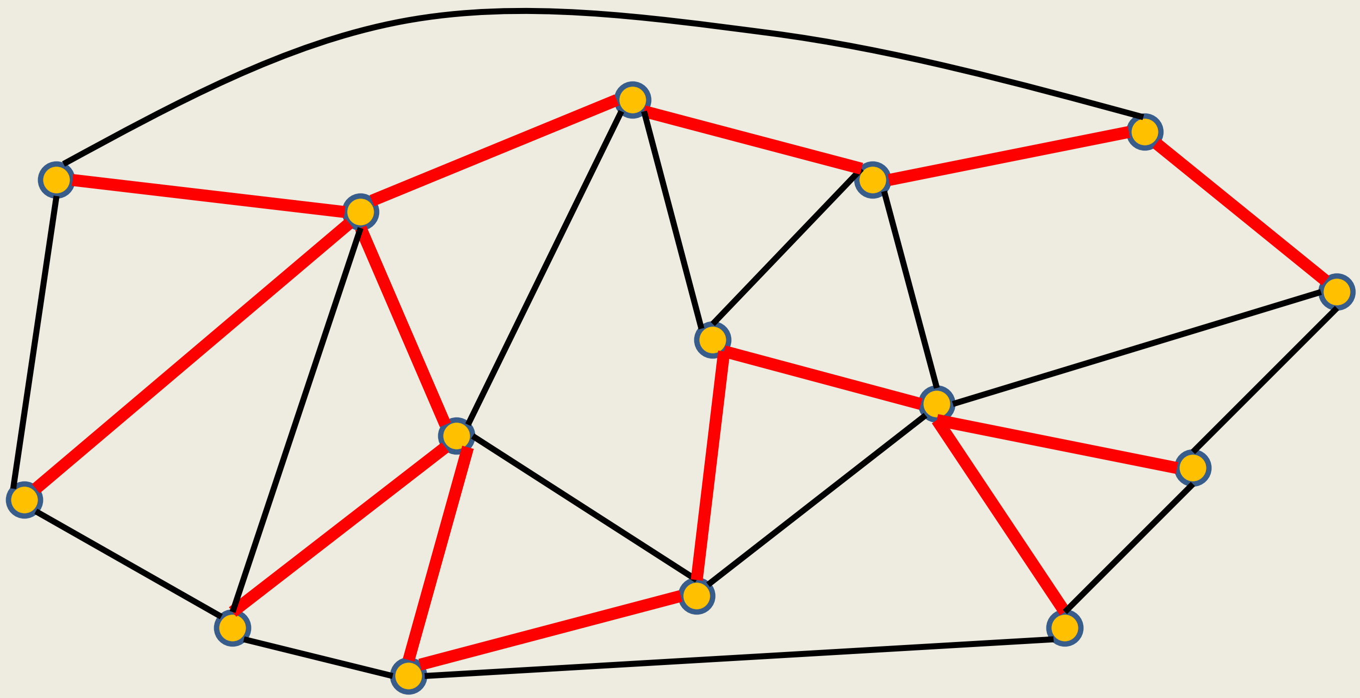
A tree

The following definitions are **equivalent**.

- An undirected graph which is **connected** but does **not have any cycle**.
- An undirected graph where each pair of vertices has **a unique path** between them.
- An undirected **connected** graph on $\boldsymbol{n}$ vertices and $\boldsymbol{n-1}$ **edges**.
- An undirected graph on $\boldsymbol{n}$ vertices and $\boldsymbol{n-1}$ **edges** and **without any cycle**.

A Spanning tree

Definition: For an undirected graph (**V**,**E**), spanning tree is a **subgraph** (**V**,**E’**), **E’** ⊆ **E** which is a tree.

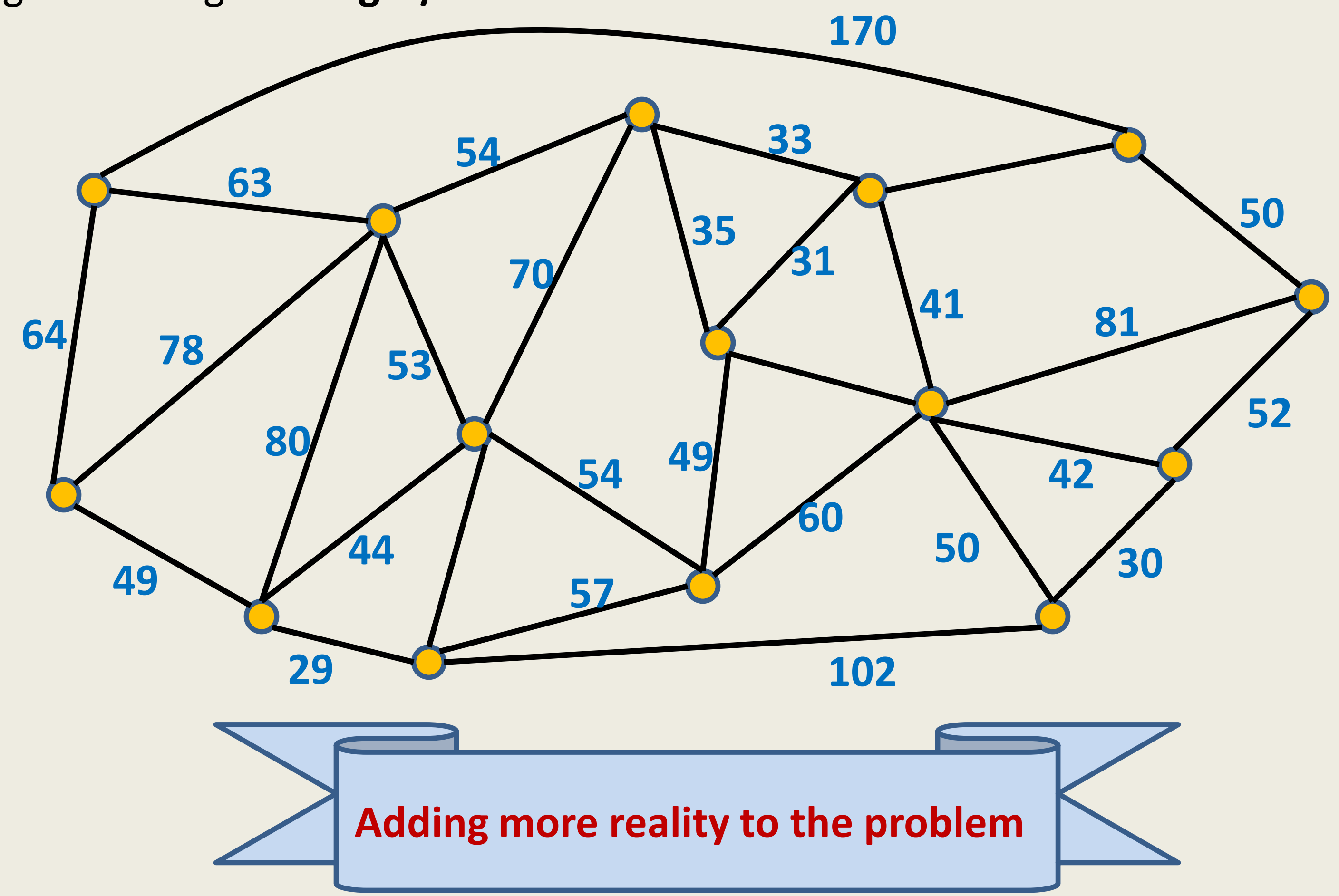


Observation: Given a spanning tree T of a graph G, adding a nontree edge e to T creates a unique cycle.

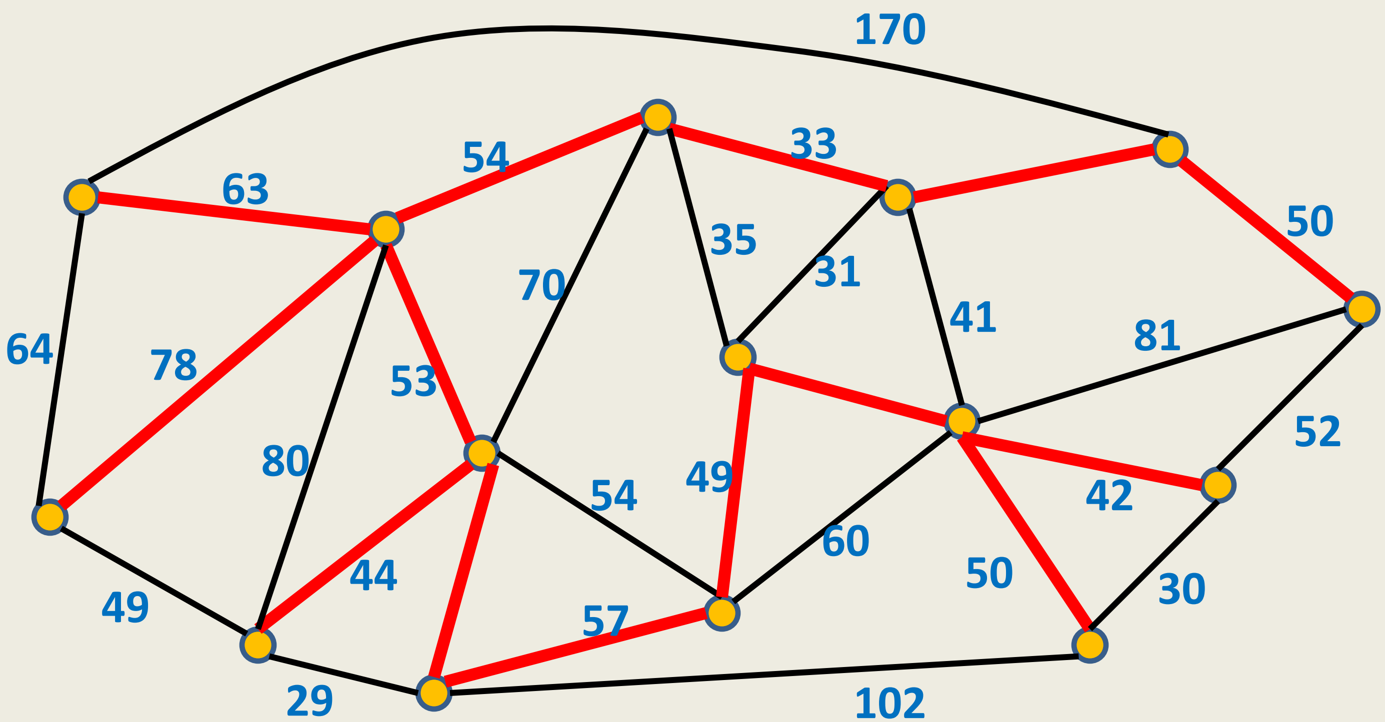
There will be total $m - n + 1$ such cycles. These are called **fundamental cycles** in G induced by the spanning tree T.

A road or telecommunication network

Assign each edge a **weight/cost**.



A road or telecommunication network



Any arbitrary spanning tree (like the one shown above) will not serve our goal☹.

We need to select the spanning tree with **least weight/cost**.

Problem 2

Minimum spanning tree

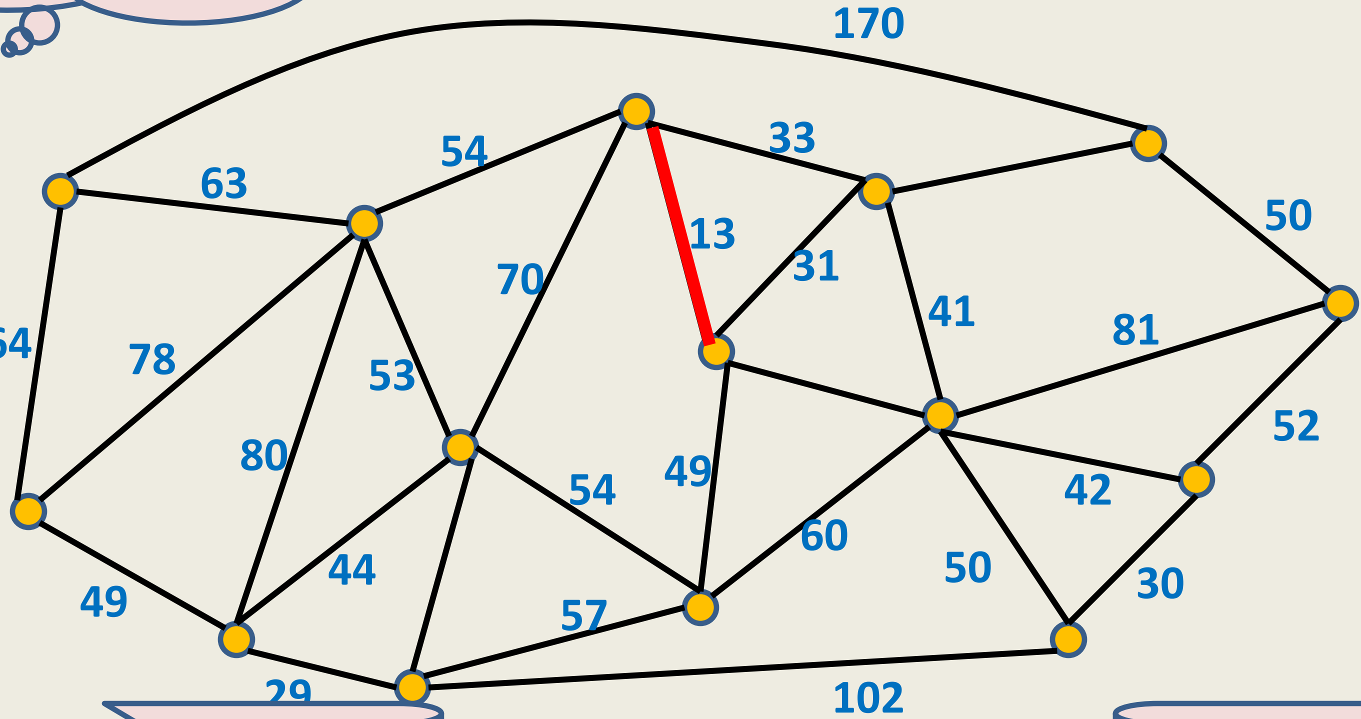
Problem Description

Input: an undirected graph **G**=(**V**,**E**) with $\mathbf{w}$: **E** → $\mathbb{R}$,

Aim: compute a **spanning tree** (**V**,**E'**), **E'** ⊆ **E** such that $\sum_{e \in E'} \mathbf{w}(e)$ is **minimum.**

How to compute a MST ?

The least weight edge
should be in MST.
But why ?



Look a … nd.
Ca …

Is there any edge for which you feel
strongly to be present in MST ?

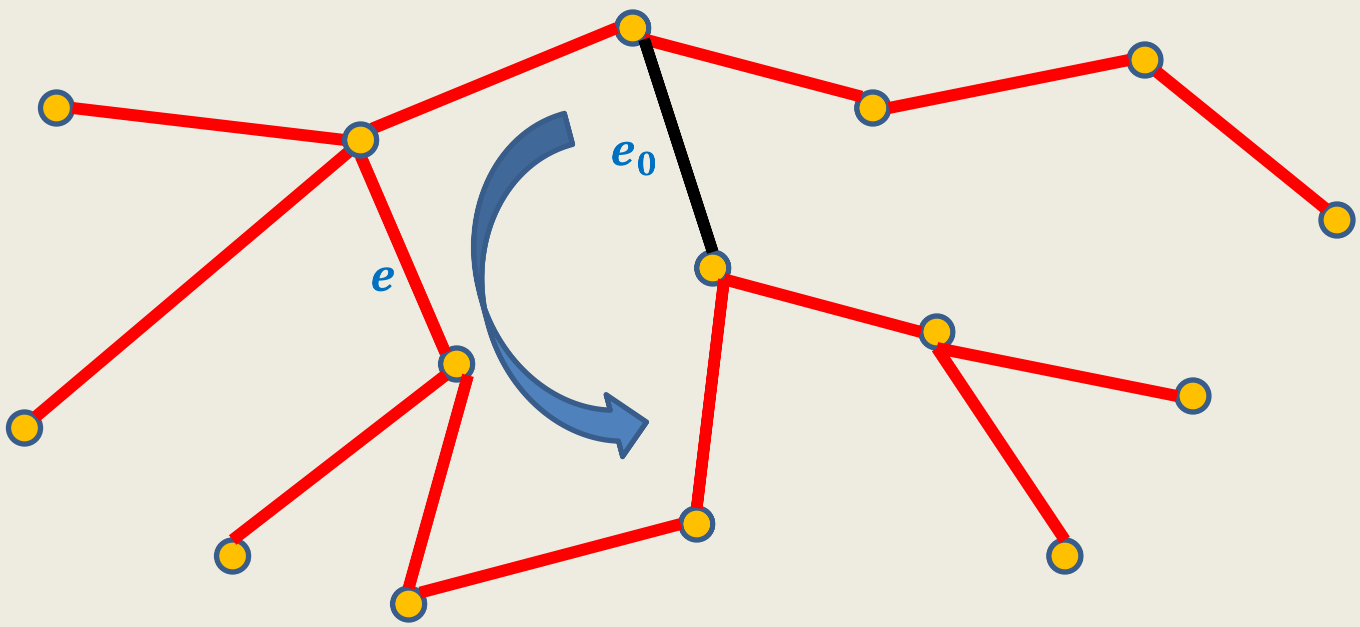
Let $\boldsymbol{e_0} \epsilon$ **E** be the edge of least weight in the given graph.

Lemma2: There is a **MST** $\boldsymbol{T}$ containing $\boldsymbol{e_0}$.

Proof: Consider any **MST** $\boldsymbol{T}$. Let $\boldsymbol{e_0} \notin \boldsymbol{T}$.

Consider the fundamental cycle $\boldsymbol{C}$ defined by $\boldsymbol{e_0}$ in $\boldsymbol{T}$.

Swap $\boldsymbol{e_0}$ with any edge $\boldsymbol{e} \epsilon\, \boldsymbol{T}$ present in $\boldsymbol{C}$.



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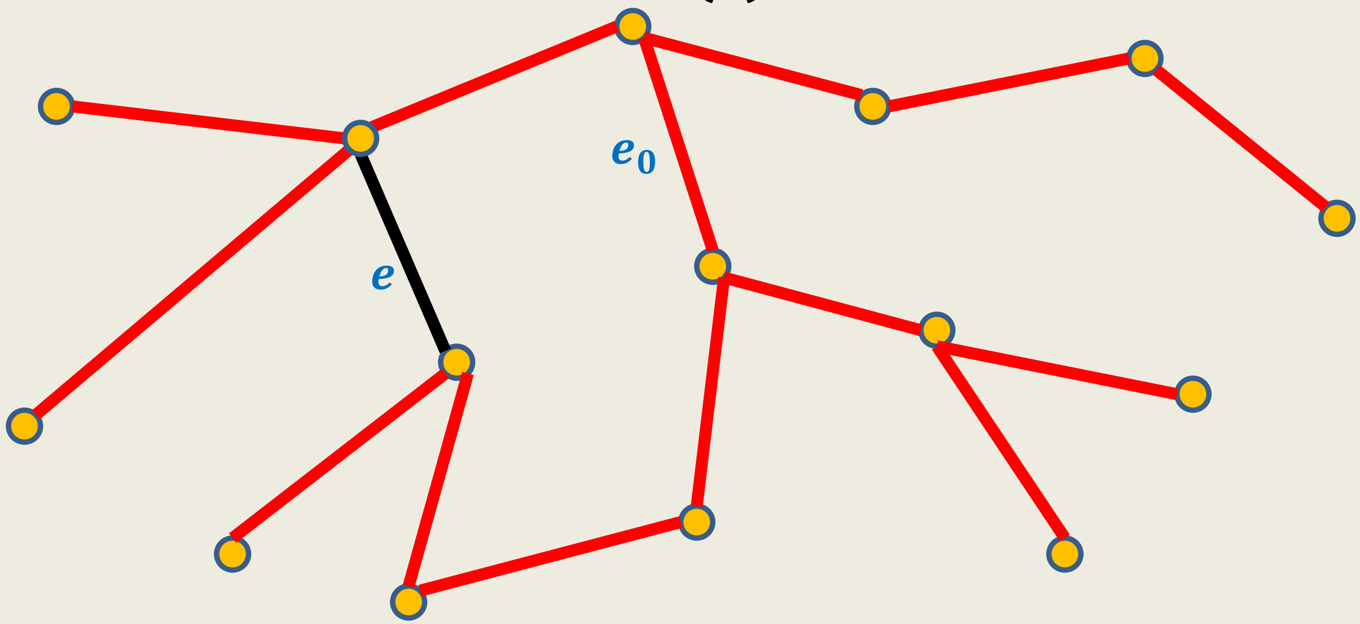
We get a spanning tree of weight ≤ $\mathbf{w}(\boldsymbol{T})$**.**



Try to translate Lemma2 to an algorithm for MST ?

with **inspiration** from the job scheduling problem ☺