Data Structures and Algorithms

(CS210A)

Semester I – **2014-15**

Lecture 32

- Algorithm for ith order statistic of a set S.

Problem definition

Given a set S of n elements and a positive integer $i \leq n$, compute ith **smallest** element from S.

Applications: As wide as that of sorting.

Trivial algorithm: Sort S

But sorting takes O(n log n) time and appears to be an overkill for this simple problem.

AIM: To design an algorithm with O(n) time complexity.

Assumption (For the sake of **neat description** and **analysis** of algorithms of this lecture):

- All elements of S are assumed to be **distinct**.

A motivational background

Though it was **intuitively appealing** to believe that there exists an **O**(n) time algorithm to compute ith smallest element, it remained a challenge for many years to design such an algorithm…

In **1972**, five well known researchers: **Blum, Floyd, Pratt, Rivest**, and **Tarjan** designed the **O**(n) time algorithm. It was designed during a **lunch break** of a conference when these five researchers sat together for the first time to solve the problem.

In this way, the problem which remained unsolved for many years got solved in less than an hour. But one should not ignore the efforts these researchers spent for years before arriving at the solution … It was their effort whose fruit got ripened in that hour ☺.

Notations

We shall now introduce some notations which will help in a neat description of the algorithm.

Notations

- S **:**

  the given set of n elements.
- e_i :

  ith **smallest** element of S.
- $S_{<x}$ **:**

  subset of S consisting of all elements **smaller than** x**.**
- $S_{>x}$ **:**

  subset of S consisting of all elements **greater than** x**.**
- **rank(**S**,**x**) :**

  1 + number of elements in S that are smaller than x**.**
- **Partition(**S**,**x**):**

  algorithm to partition S into $S_{<x}$ and $S_{>x}$;

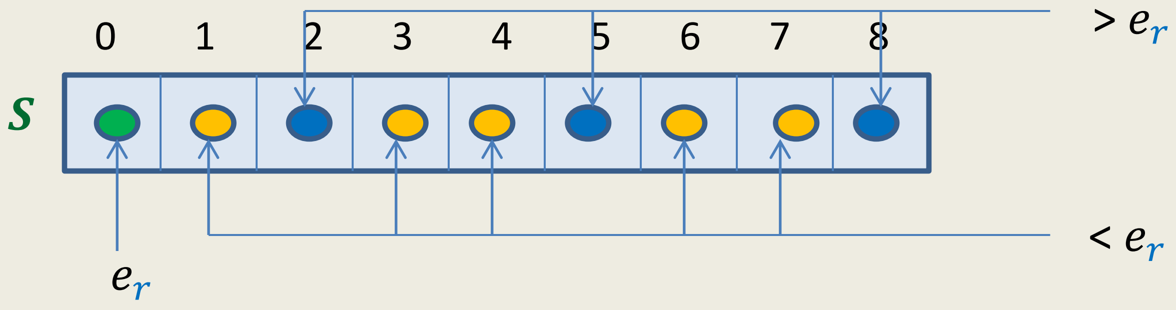
  this algorithm returns **(**$S_{<x}$, $S_{>x}$, r**)** where r**=rank(**S**,**x**).**

Algorithm 1

(algorithm derived from **QuickSort**)

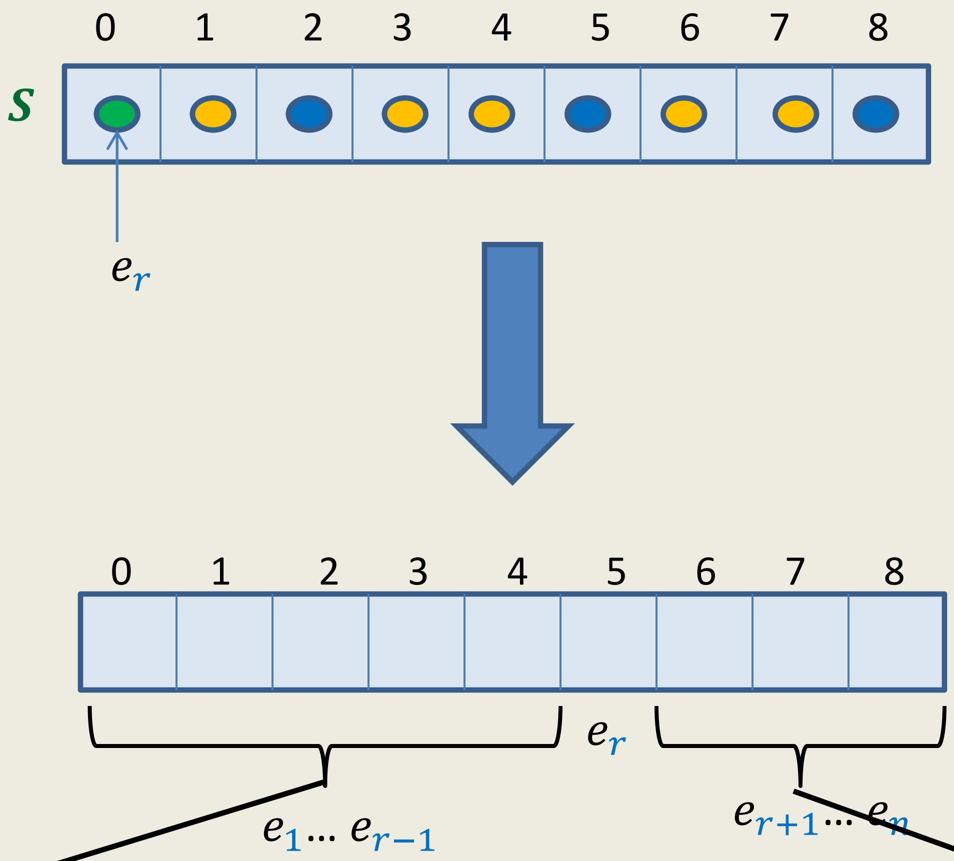
QuickSelect(S,i)

Quick Sort



What happens during
Partition(S,0,8)

Quick Sort



If $i > r$ we can discard these element.

If $i < r$ we can discard these elements.

Pseudocode for QuickSelect(S,i)

```
QuickSelect($S$,$i$)
{       Pick an element $x$ from $S$;
        ($S_{<x}$, $S_{>x}$, $r$)← Partition($S$,$x$);
         If($i$ = $r$) return $x$;
         Else If ($i$< $r$)
                        QuickSelect($S_{<x}$,$i$)
             Else
                        QuickSelect($S_{>x}$,$i - r$);
}
```

Worst case time complexity : O(n^2)

Average case time complexity: O(n) — Analysis is simpler than Quick Sort.

Towards worst case O(n) time algorithm …

Key ideas

- **Inspiration** from some recurrences.
- Concept of **approximate median**
- Learning from **QuickSelect**($\boldsymbol{S}$,$\boldsymbol{i}$)

Learning from recurrences

Question: what is the solution of recurrence **T**(n) = cn **+ T**($9n/10$) ?

Answer: O(n).

Sketch (by gradual unfolding):

T(n) = cn + c $9n/10$ + c $81\ n/100$ + …

= $cn[1$ + $9/10$ + $81/100$ + …$]$

= O(n)

Lesson learnt: Solution for **T**(n) = cn **+ T**(an) is **O(n) if** $0 < a < 1$.

Learning from recurrences

Question: what is the solution of recurrence **T**(n) = cn **+ T**($n/6$) + **T**($5n/7$) ?

Answer: O(n).

Sketch: (by induction)

Assertion: **T**(n) ≤ $c_1 n$.

Induction step: T(n) = cn **+ T**($n/6$) + **T**($5n/7$)

$\leq cn + \frac{37}{42} c_1 n$

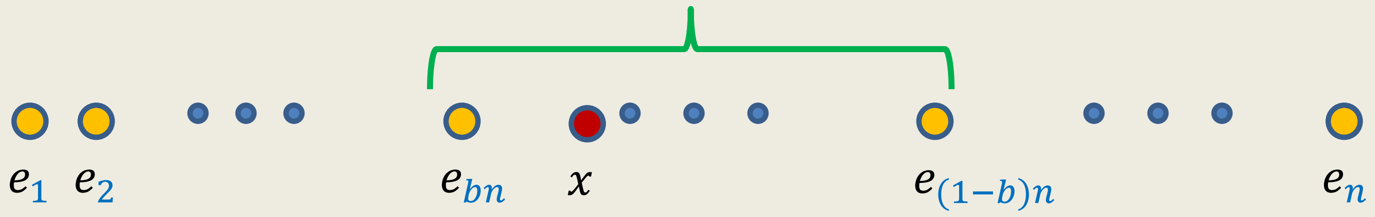
$\leq c_1 n$ if $c_1 \geq \frac{42}{5}$ c

Lesson learnt from the above proof ?

: Solution for **T**(n) = cn **+ T**(an) + **T**(dn) is **O(n) if** $a + d < 1$.

Concept of **approximate median**

Definition: Given a constant $0 < \boldsymbol{b} \leq \mathbf{1/2}$,
an element $\boldsymbol{x} \in \boldsymbol{S}$ is said to be $\boldsymbol{b}$-approximate median of $\boldsymbol{S}$
if **rank(**$\boldsymbol{x}, \boldsymbol{S}$**)** is in the range $[\boldsymbol{bn}, (1 - \boldsymbol{b})\boldsymbol{n}]$.



Learning from QuickSelect(S,i)

```
QuickSelect($S$,$i$)
{       Pick an element $x$ from $S$;                            }
        ($S_{<x}$, $S_{>x}$, $r$) ← Partition($S$,$x$);          } O($n$)
         If($i$ = $r$) return $x$;                               }
         Else If ($i$< $r$)                                      }
                     QuickSelect($S_{<x}$,$i$)                    } T( $(1-b)n$)
              Else                                               }
                     QuickSelect($S_{>x}$,$i-r$);                 }
}
```

Question: If x is a b-approximate median of S, what is time complexity of the algorithm ?

Answer: T(n) = cn + T($(1-b)n$)

= O(n) **[Learning from Recurrence of type 1]**

Algorithm 2

Select($\boldsymbol{S}$,$\boldsymbol{i}$)

(A **linear time** algorithm)

Overview of the algorithm

```
Select($S$,$i$)
{      Compute a $b$-approximate median, say $x$, of $S$;          } O($n$)+ T($dn$)
       ($S_{<x}$, $S_{>x}$, $r$) ← Partition($S$,$x$);            } O($n$)
       If($i$ = $r$) return $x$;                                   ┐
       Else If ($i$< $r$)                                          │
                 Select($S_{<x}$,$i$)                              ├ T( $(1-b)n$)
            Else                                                   │
                 Select($S_{>x}$,$i-r$);                           ┘
}
```

Observation: If we can compute b-approximate median in O(n) time, we get O(n) time algo.

But that appears too much to expect from us. Isn't it ?

So what to do ☹ ?

Hint: Can you refine the above observation using **Lessons you learnt from Recurrence of type II** ?

Observation: If we can compute b-approximate median in O(n)+ T(dn) time for $d+(1-b) < 1$, the time complexity of the algorithm will still be O(n).

Spend some time on this observation to infer what it hints at.

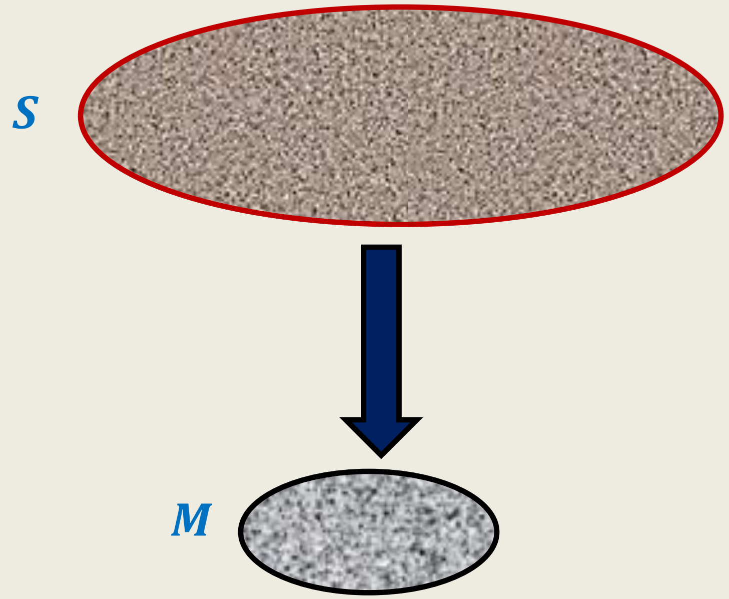
Overview of the algorithm

```
Select($S$,$i$)
{       Compute a $b$-approximate median, say $x$, of $S$;          } O($n$)+ T($dn$)
        ($S_{<x}$, $S_{>x}$, $r$)← Partition($S$,$x$);              } O($n$)
        If($i$ = $r$) return $x$;
        Else If ($i$< $r$)
                        Select($S_{<x}$,$i$)                         } T( $(1-b)n$)
                Else
                        Select($S_{>x}$,$i-r$);
}
```

Hint: use **Lessons you learnt from Recurrence of type II** ?

Observation: If d+$(1-b)$ < 1, the time complexity will still be **O(n).**

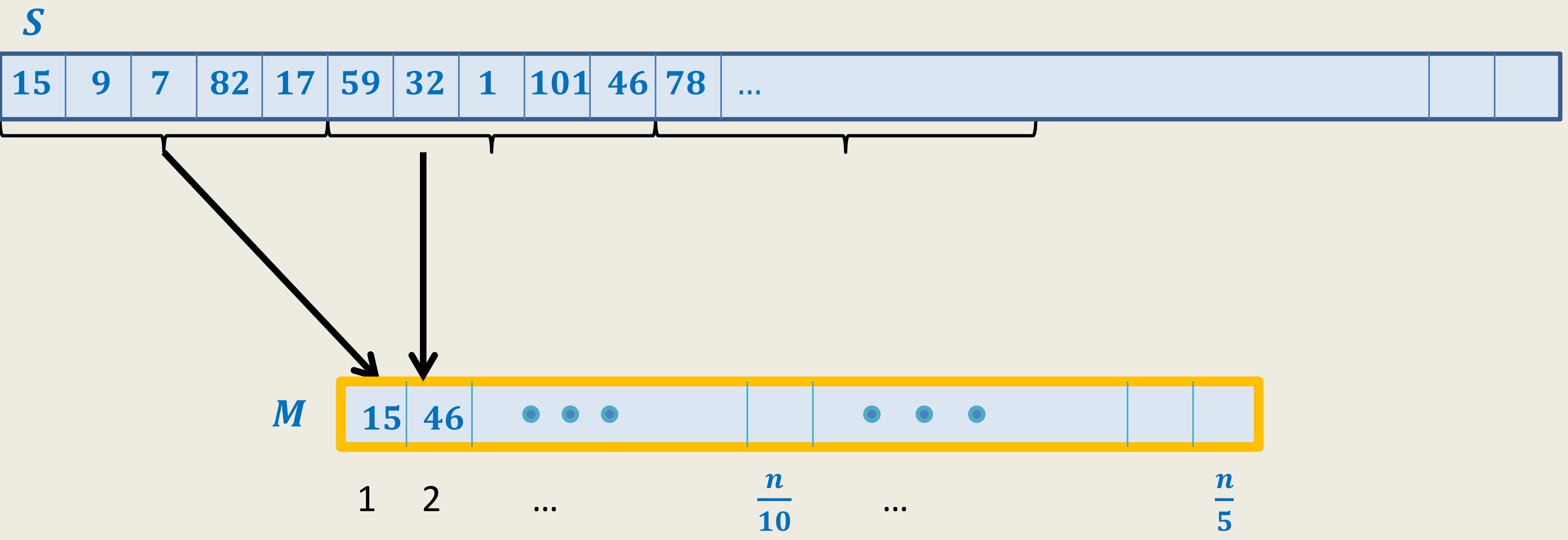
AIM: How to compute a b-approximate median of S in **O**(n)**+ T**(dn) time with d+$(1-b)$ < 1?



Question: Can we form a set M of size dn such that

exact median of M is b-**approximate** median of S?

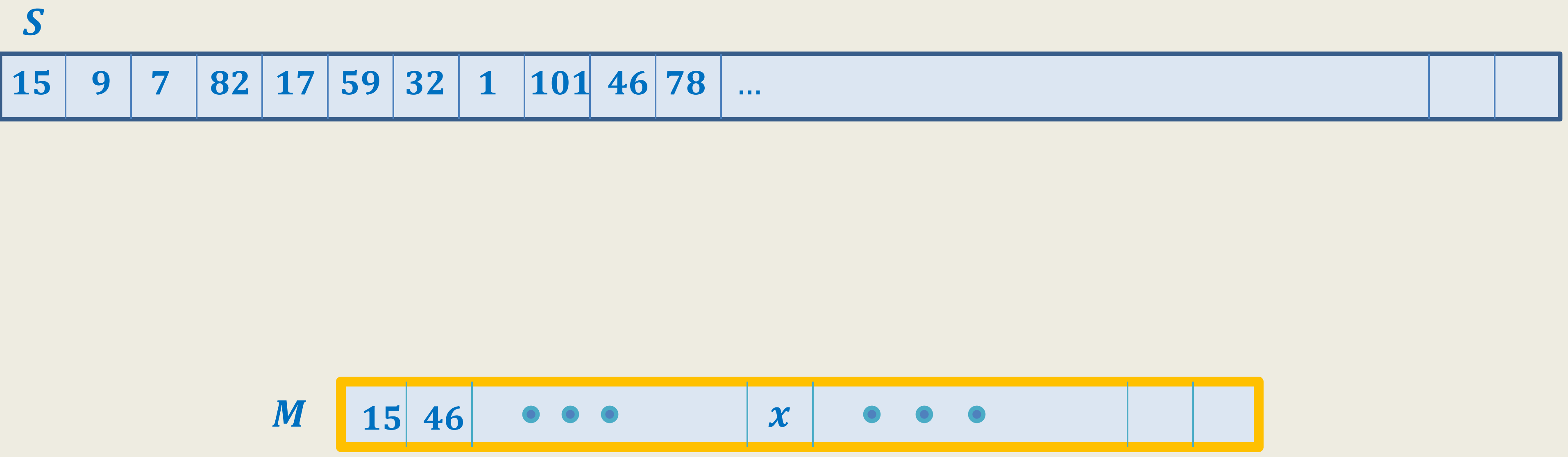
Forming the subset M



- Divide S into **groups** of 5 elements;
- Compute median of each group by sorting;

$\left.\right\}$ O(n)

Forming the subset M

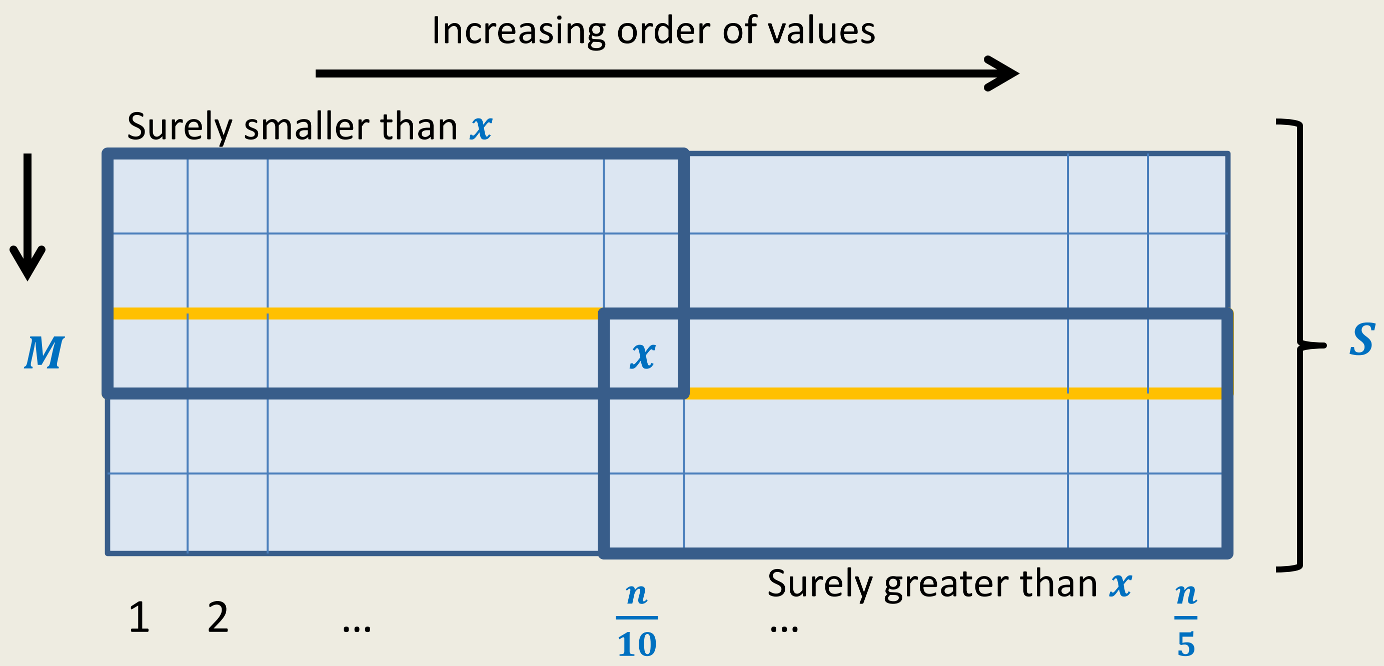


- Divide S into **groups** of 5 elements;
- Compute median of each group by sorting;
- Let M be the set of medians;
- Let x be median of M.

What can we say about rank of x in S ?

Spend some time to answer this question before moving ahead.

Forming the subset M



x is $\left(\frac{3n}{10}\right)$ −approximate median of S.

Time required to form M : O(n)

Pseudocode for Select(S,i)

```
Select($S$,$i$)
        $M \leftarrow \emptyset$;
        Divide $S$ into **groups** of 5 elements;                                    O(n)
        **Sort** each group and **add its median** to $M$;
        $x \leftarrow$ Select($M$,|$M$|/2);                                  ⇐ T(n/5)
        ($S_{<x}$, $S_{>x}$, $r$) $\leftarrow$ Partition($S$,$x$);          ⇐ O(n)
        If($i$ = $r$) return $x$;
        Else If ($i$< $r$)
                        Select($S_{<x}$,$i$)                                      T(7n/10)
            Else
                        Select($S_{>x}$,$i - r$);
```

Analysis

T(n) = cn + T(n/5) + **T(7n/10)**

= O(n) **[Learning from Recurrence of type II]**

Theorem: Given any S of **n** elements, we can compute ith smallest element from S in O(**n**) worst case time.

Exercises

(Attempting these exercises will give you a better insight into the algorithm.)

- What is magical about number 5 in the algorithm ?
- What if we divide the set $\boldsymbol{S}$ into groups of size 3 ?
- What if we divide the set $\boldsymbol{S}$ into groups of size 7 ?
- What if we divide the set $\boldsymbol{S}$ into groups of even size (e.g. 4 or 6) ?