Data Structures and Algorithms

(CS210A)

Semester I – **2014-15**

Lecture 31

- Magical applications of Binary trees -II

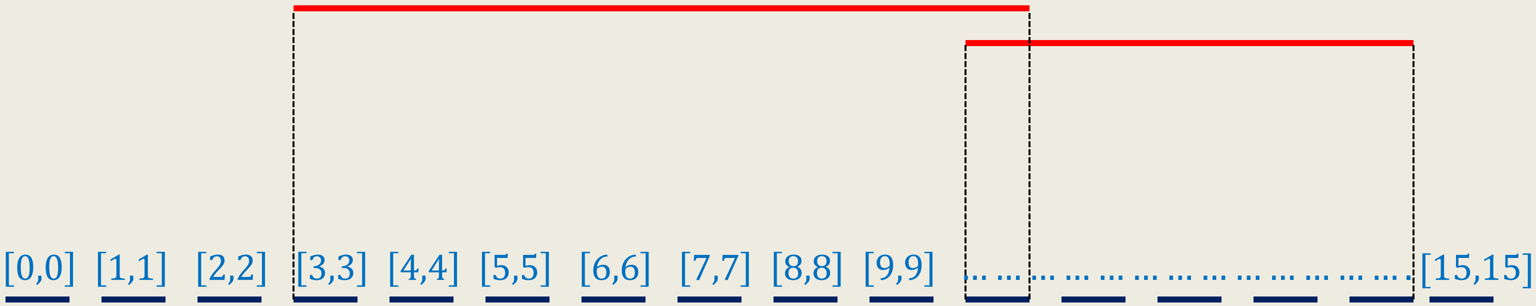
RECAP OF LAST LECTURE

Intervals

S = {[$\boldsymbol{i}, \boldsymbol{j}$], $0 \le i \le j < n$}

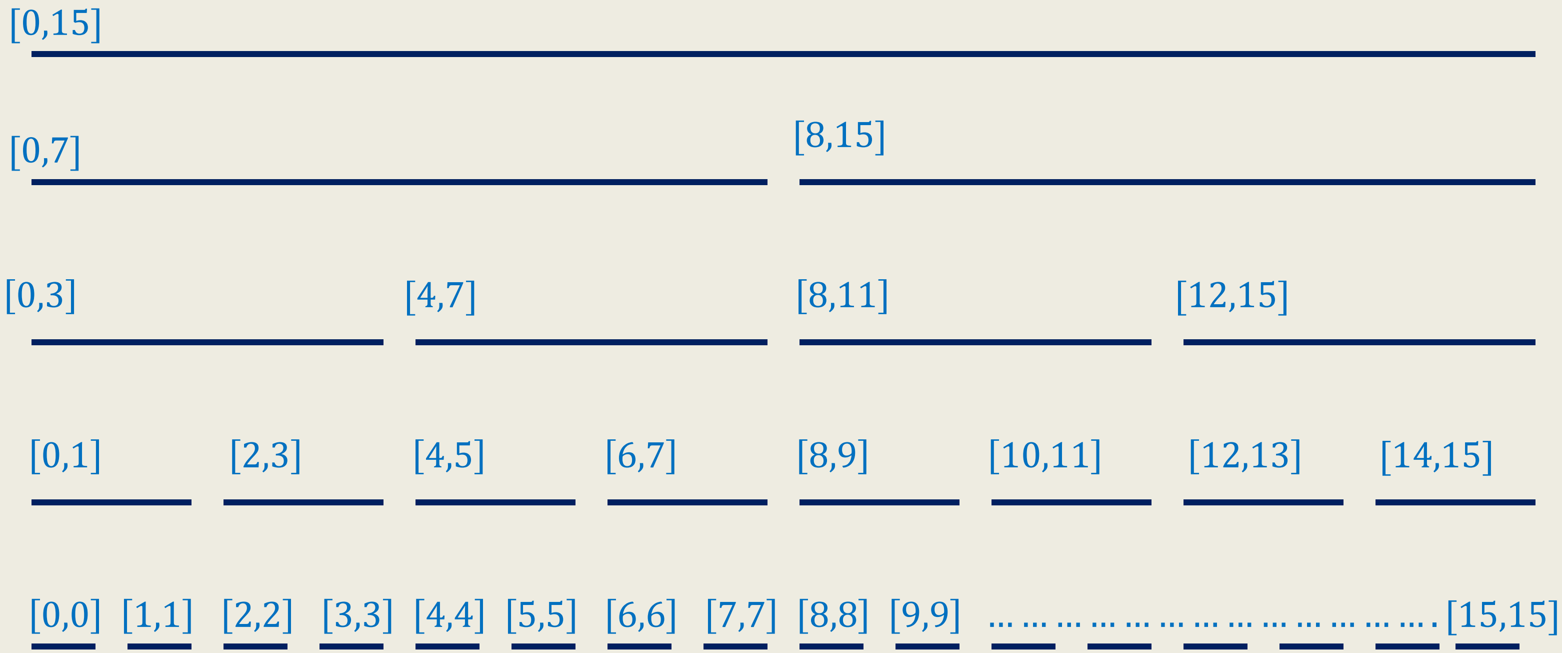
Question: Can we have a small set **X**⊂**S** of **intervals** s.t.

every interval in **S** can be expressed as a union of a few **intervals** from **X** ?



Answer: yes☺

Hierarchy of intervals



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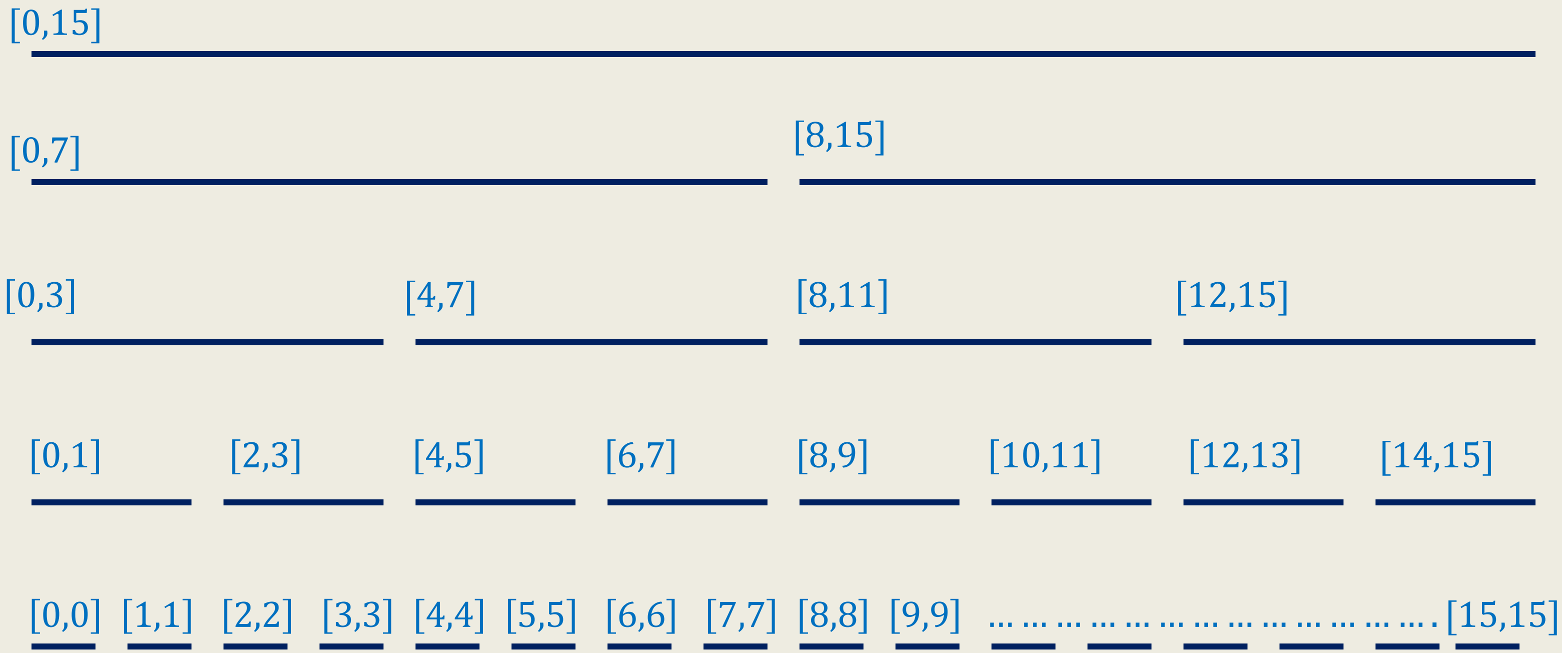

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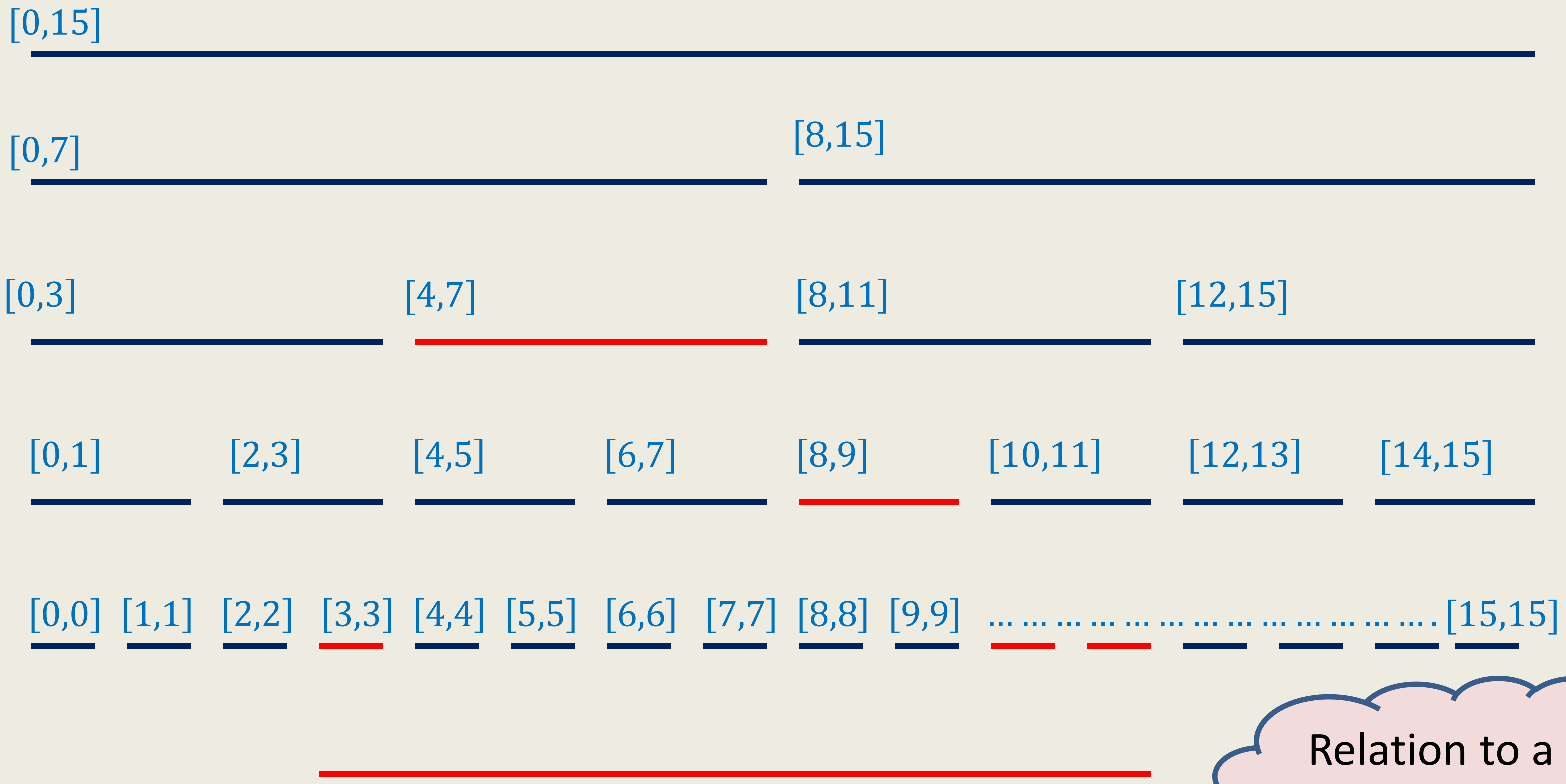
Observation: There are $2n$ intervals such that

any interval $[i, j]$ can be expressed as **union** of **O**(log n) basic intervals ☺

Hierarchy of intervals



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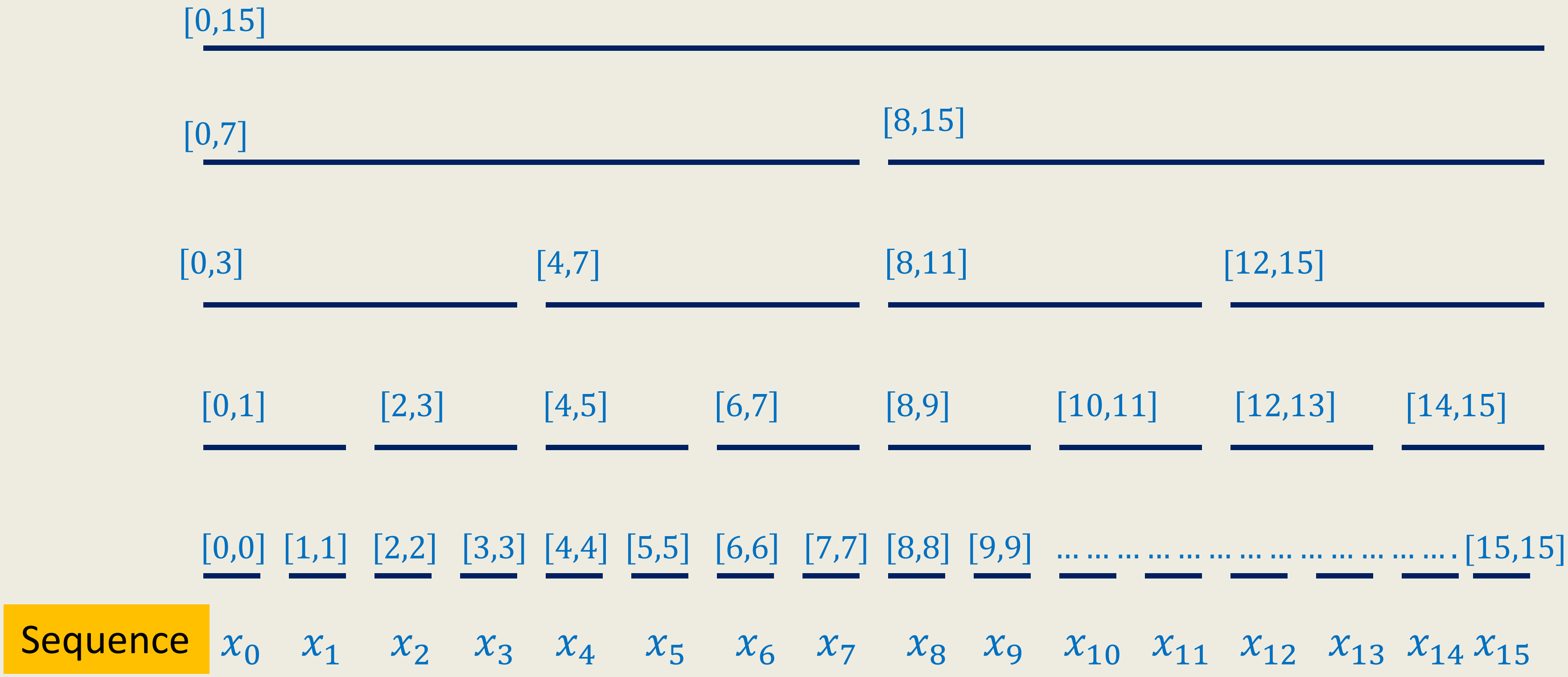
Relation to a **sequence** ?

Observation: There are $2n$ intervals such that any interval $[i, j]$ can be expressed as **union** of **O**(log n) basic intervals ☺

Which data structure emerges ?



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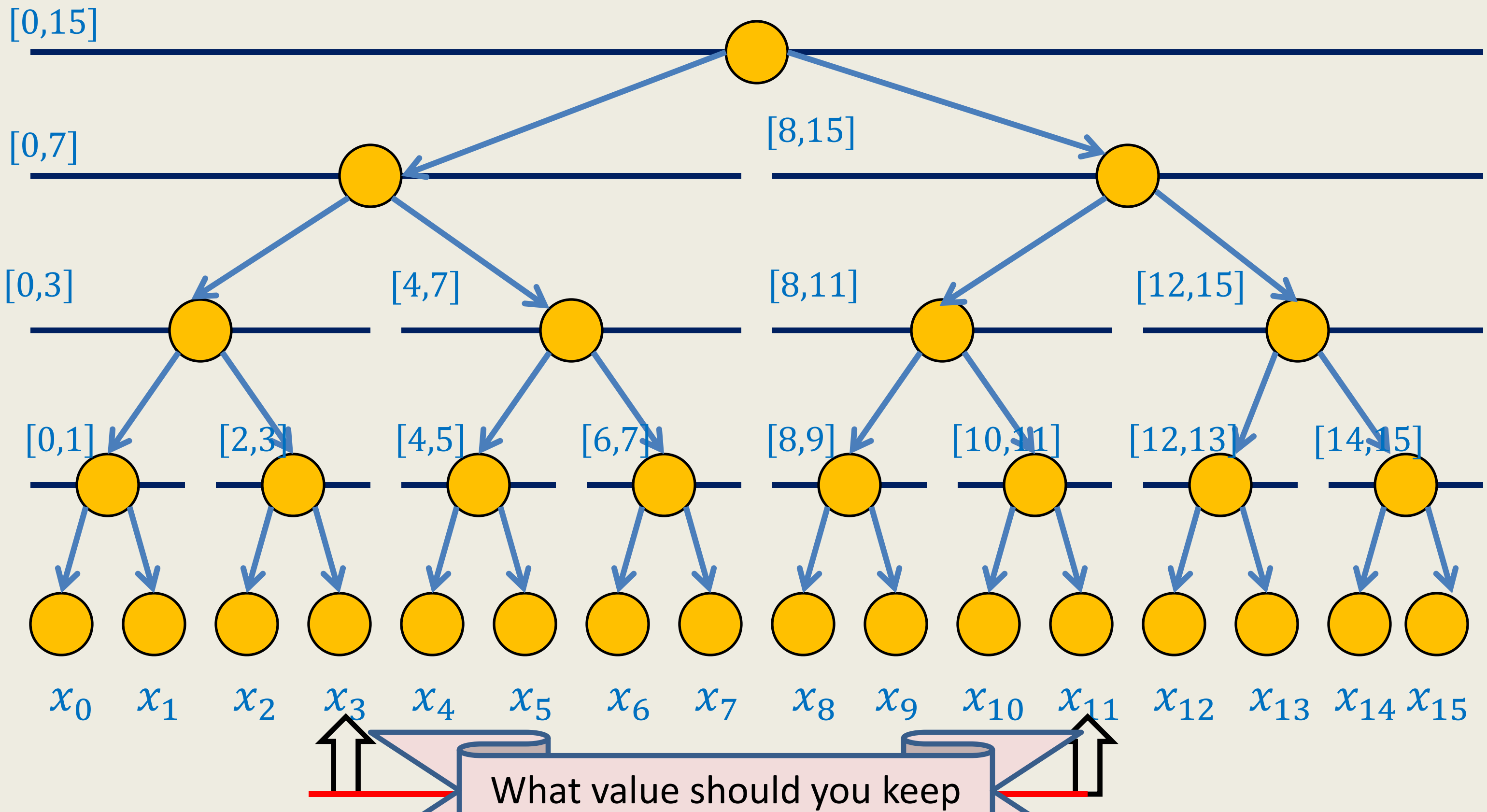
Sequence x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13} x_{14} x_{15}

A Binary tree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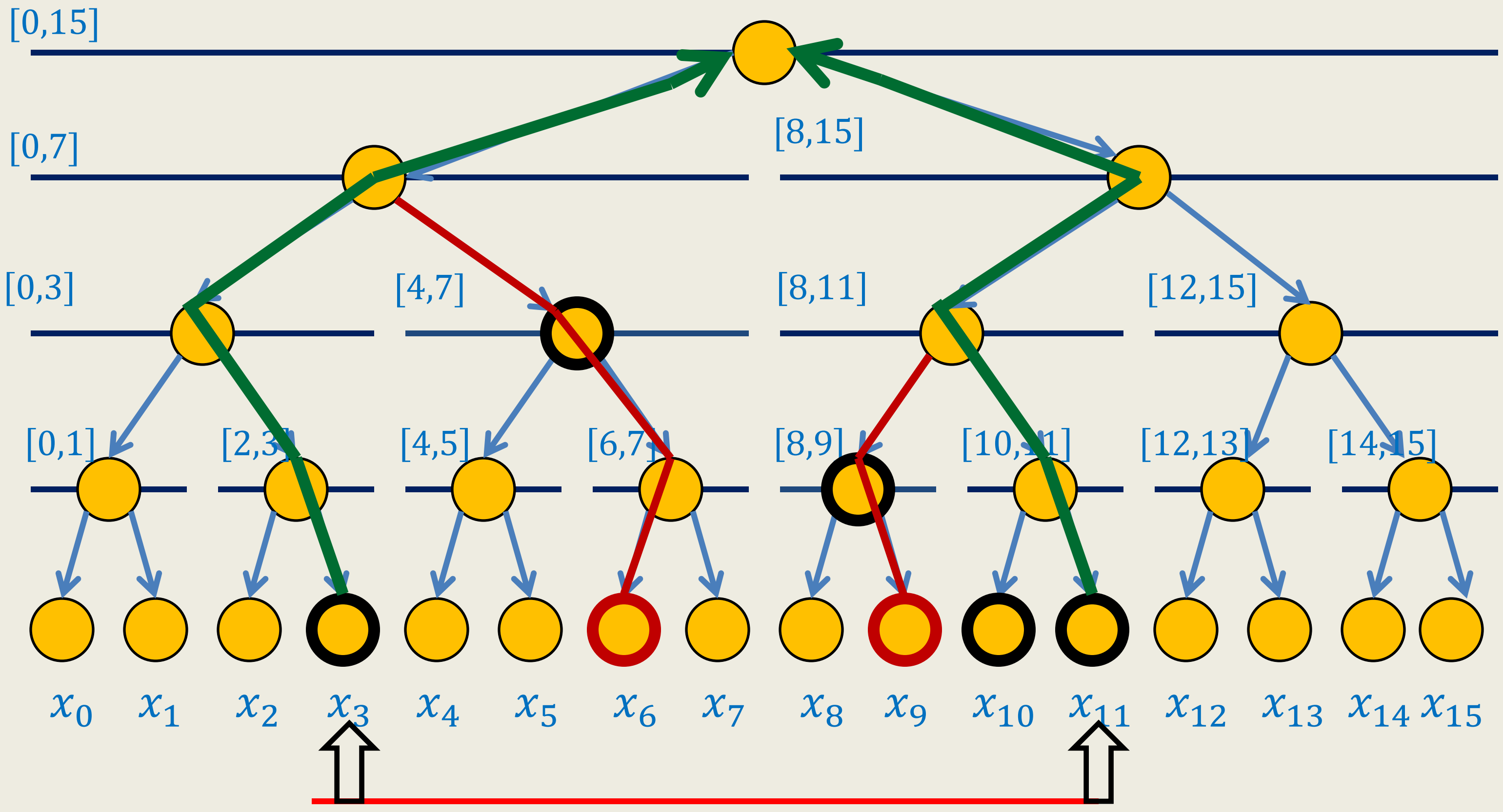
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13} x_{14} x_{15}

What value should you keep in **internal nodes** ?

How to perform Operation on an interval ?



How to perform Operation on an interval ?

Problem 2

Dynamic Range-minima

Dynamic Range Minima Problem

Given an initial sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ of numbers, maintain a compact data structure to perform the following operations efficiently for any $0 \le \boldsymbol{i} < \boldsymbol{j} < \boldsymbol{n}$.

- **ReportMin**($\boldsymbol{i}$, $\boldsymbol{j}$):

  Report the minimum element from $\{x_k \mid \text{for each } \boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}\}$

- **Update**($\boldsymbol{i}$, **a**):

  a becomes the new value of x_i.

Example:

Let the initial sequence be **S** = $\prec$ 14, 12, 3, 49, 4, 21, 322, -40 $\succ$

ReportMin($\mathbf{1}$, $\mathbf{5}$) returns **3**

ReportMin($\mathbf{0}$, $\mathbf{3}$) returns **3**

Update($\mathbf{2}$, **19**) update **S** to $\prec$ 14, 12, **19**, 49, 4, 21, 322, -40 $\succ$

ReportMin($\mathbf{1}$, $\mathbf{5}$) returns **4**

ReportMin($\mathbf{0}$, $\mathbf{3}$) returns **12**

Dynamic Range Minima Problem

Given an initial sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ of numbers, maintain a compact data structure to perform the following operations efficiently for any $0 \leq \boldsymbol{i} < \boldsymbol{j} < \boldsymbol{n}$.

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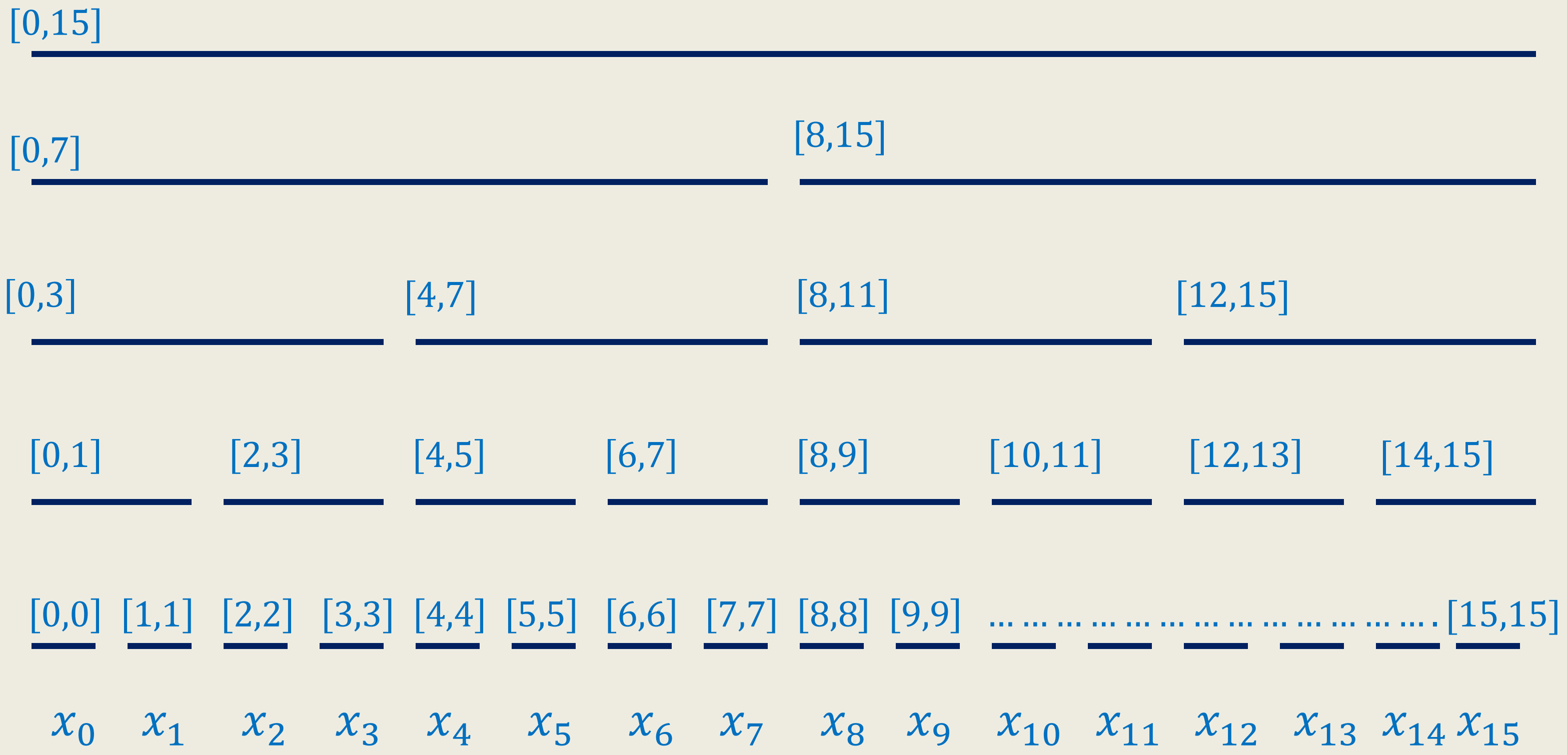
- **Update**($\boldsymbol{i}$, **a**):

  a becomes the new value of x_i.

AIM:

- **O**($\boldsymbol{n}$) size data structure.
- **ReportMin**($\boldsymbol{i}$, $\boldsymbol{j}$) in **O**(log $\boldsymbol{n}$) time.
- **Update**($\boldsymbol{i}$, **a**) in **O**(log $\boldsymbol{n}$) time.

Hierarchy of intervals



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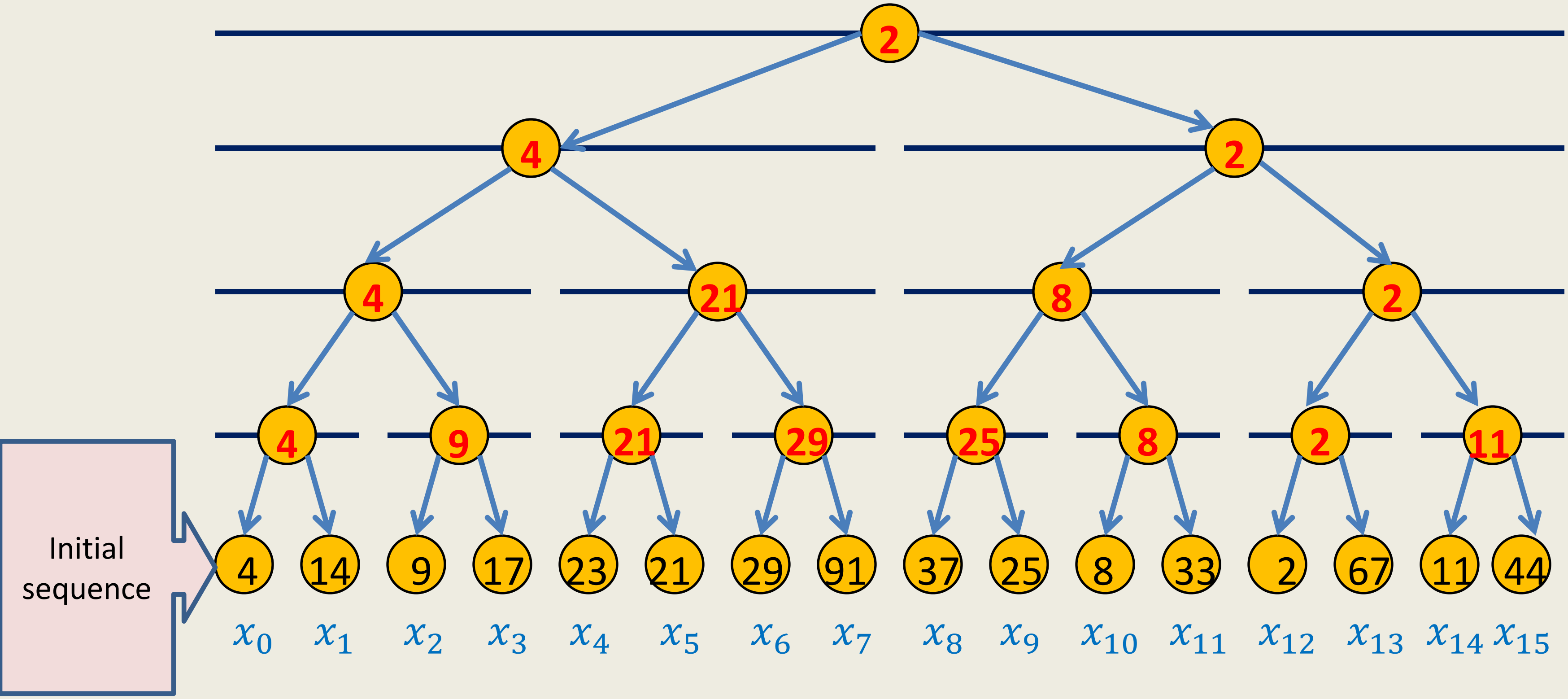
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13} x_{14} x_{15}

Observation: There are $2\boldsymbol{n}$ intervals such that any interval $[\boldsymbol{i}, \boldsymbol{j}]$ can be expressed as **union** of **O**(log $\boldsymbol{n}$) basic intervals ☺

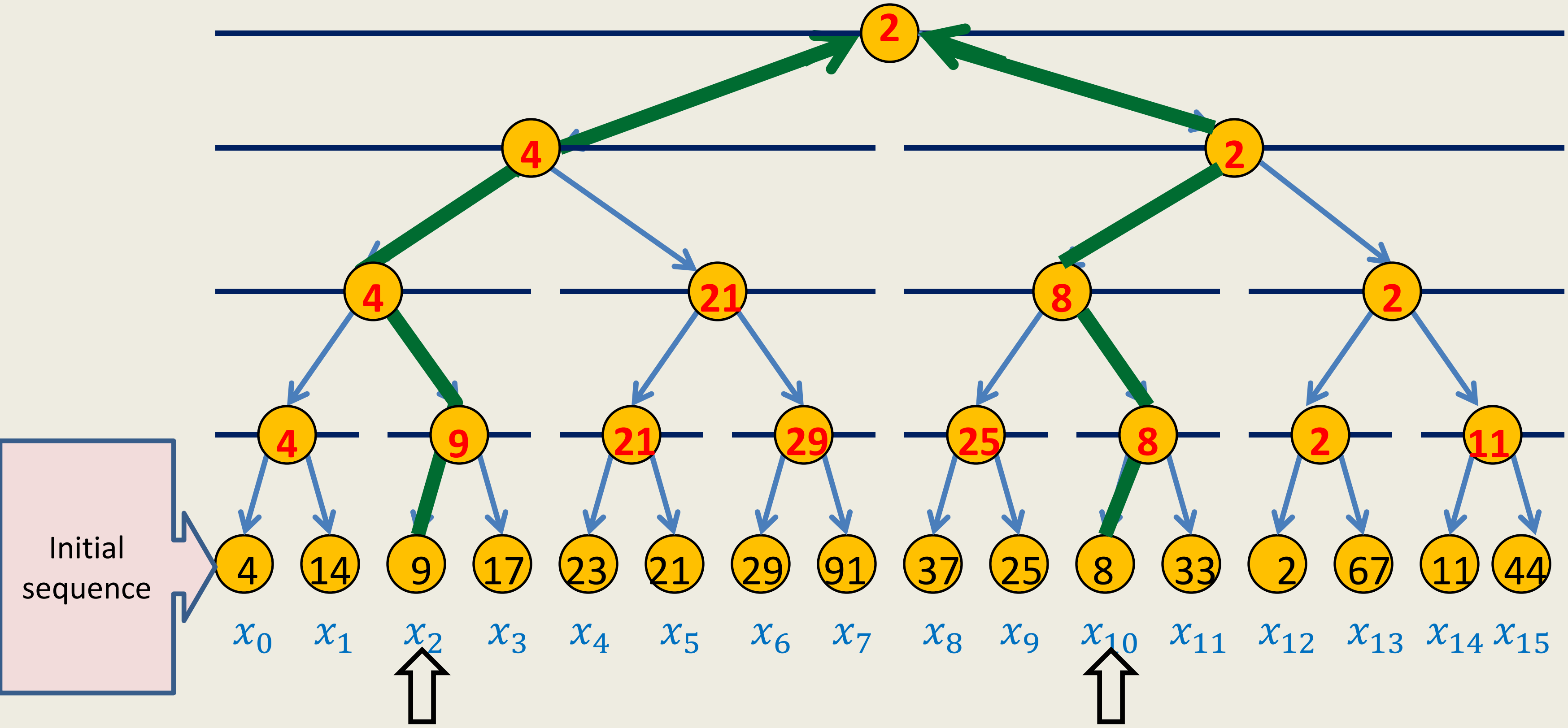
Data structure for dynamic range minima



Question: What should be stored in an internal node **v** ?

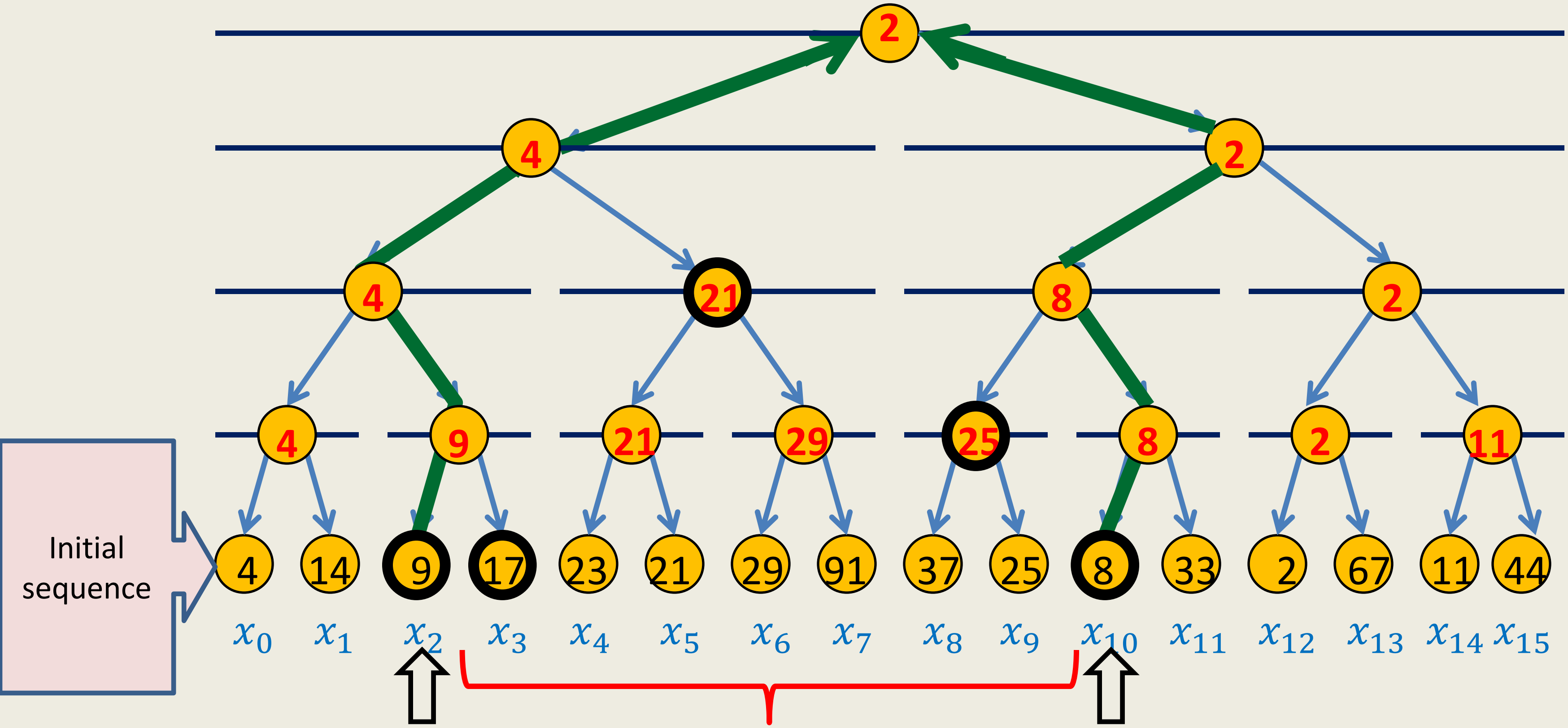
Answer: **minimum** value stored in **subtree**(**v**).

Data structure for dynamic range minima



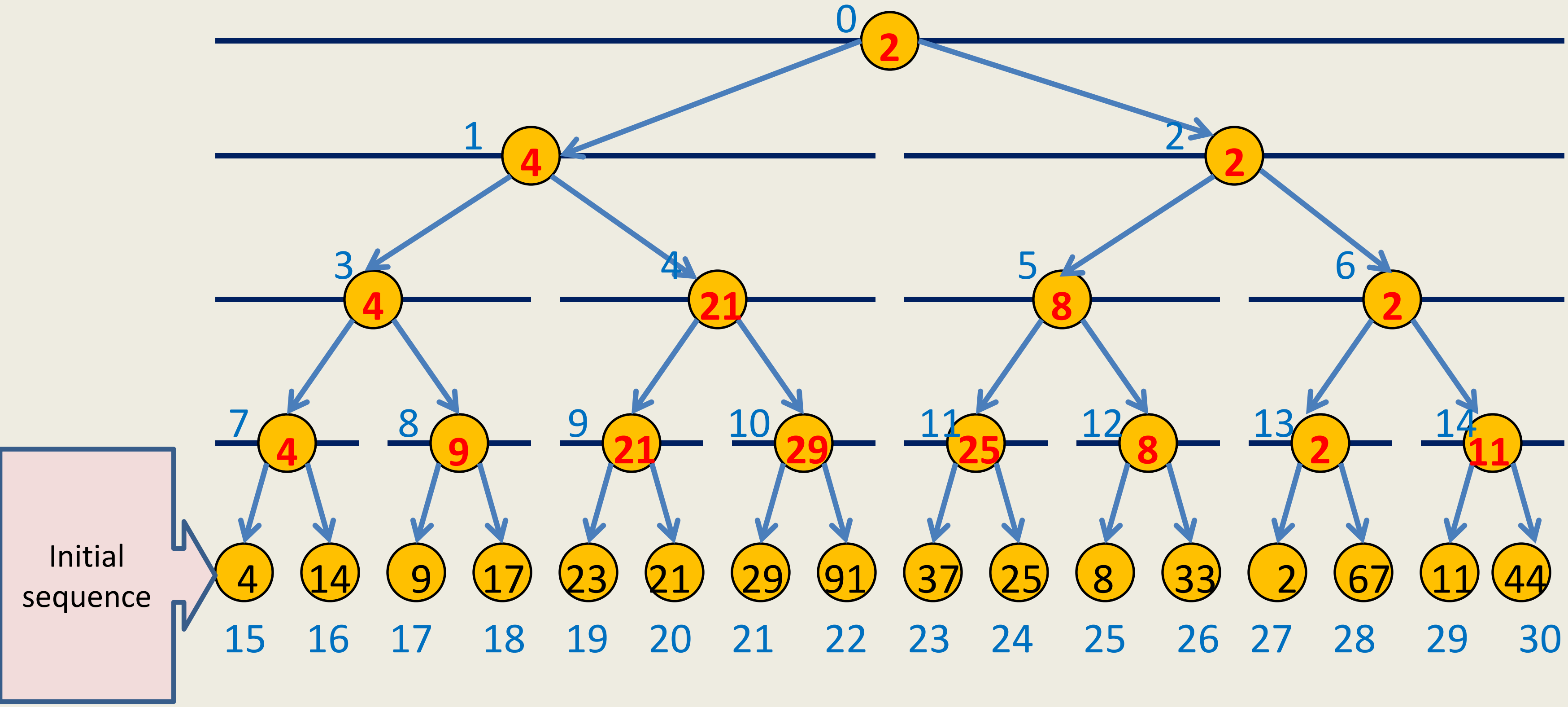
How to do Report-Min(2,10) ?

Data structure for dynamic range minima



How to do Report-Min(2,10) ?

Data structure for dynamic range minima



Data structure: An array **A** of size 2**n**-1.

Copy the sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ into **A**$[n-1]$…**A**$[2n-2]$

Leaf node corresponding to x_i = **A**$[(n-1)+i]$

How to check if a node is left child or right child of its parent ?

(if index of the node is odd, then the node is left child, else the node is right child)

Update(i, a)

```
Update($i, a$)
    $i \leftarrow (n-1) + i$ ;
    A[$i$]$\leftarrow$ $a$ ;
    $i$ $\leftarrow$ $\lfloor (i-1)/2 \rfloor$ ;
    While(          ??          )
    {
            Homework
    }
```

Report-Min(i,j)

```
Report-Min($i$,$j$)
    $i \leftarrow (n-1)+i$ ;
    $j \leftarrow (n-1)+j$ ;
    min ← A($i$);
    If ($j$ > $i$)
    {       If (A($j$)< min)   min ← A($j$);
            While( $\lfloor (i-1)/2 \rfloor$ <> $\lfloor (j-1)/2 \rfloor$ )
            {
                If($i$%2=1 and A($i+1$)< min)   min ← A($i+1$);
                If($j$%2=0 and A($j-1$)< min)   min ← A($j-1$);
                $i \leftarrow \lfloor (i-1)/2 \rfloor$ ;
                $j \leftarrow \lfloor (i-1)/2 \rfloor$ ;
            }
    }
    return min;
```

Proof of correctness

Let T be the tree data structure for **Dynamic Range-minima** problem.

Let **u** be any node in **T**.

Question:

What can we say about **value**(**u**) after a series of operations ?

Answer:

After every operation:

value(**u**) is **minimum** among all values stored in the leaf nodes of **subtree**(**u**).

Another interesting problem on sequences

Practice Problem

Given an initial sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ of n numbers,

maintain a compact data structure to perform the following operations efficiently :

- **Report_min**($\boldsymbol{i}$, $\boldsymbol{j}$):

  Report the minimum element from $\{x_i, \ldots x\_j\}$.

- **Multi-Increment**($\boldsymbol{i}$, $\boldsymbol{j}$, **Δ**):

  Add **Δ** to each x_k for each $\boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}$

Example:

Let the initial sequence be **S** = $\prec$ 14, 12, 3, 12, 111, 51, 321, -40 $\succ$

Report_min(**1** , **4**):

returns 3

Multi-Increment(2,6,10):

S becomes $\prec$ 14, 12, **13**, **22**, **121**, **61**, **331**, -40 $\succ$

Report_min(**1** , **4**):

returns 12

An challenging problem on sequences

For winter vacation

(not for the exam)

$\ast$ Problem

Given an initial sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ of n numbers,
maintain a compact data structure to perform the following operations efficiently :

- **Report_min**($\boldsymbol{i}$, $\boldsymbol{j}$):

  Report the minimum element from $\{x_i, \ldots x\_j\}$.

- **Multi-Increment**($\boldsymbol{i}$, $\boldsymbol{j}$, **Δ**):

  Add **Δ** to each x_k for each $\boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}$

- **Rotate**($\boldsymbol{i}$, $\boldsymbol{j}$):

  $x_i \leftrightarrow x_j$, $x_{i+1} \leftrightarrow x_{j-1}$, ….

Example:

Let the initial sequence be **S** = $\prec$ 14, 12, 23, 19, 111, 51, 321, -40 $\succ$

After **Rotate**(1,6), **S** becomes

$\prec$ 14, **321**, **51**, **111**, **19**, **23**, **12**, -40 $\succ$

Problem 4

A data structure for sets

Sets under operations

Given: a collection of n singleton sets $\{0\}$, $\{1\}$, $\{2\}$, … $\{n-1\}$

Aim: a compact data structure to perform

- Union(i, j):

  Unite the two sets containing i and j.
- Same_sets(i, j):

  Determine if i and j belong to the same set.

Trivial Solution 1

Keep an array **Label[]** such that

Label[i]=**Label**[j] if and only if i and j belong to the same set.

➔ Same_sets(i, j):

check if **Label**[i]=**Label**[j] ?

O(1) time

➔ Union(i, j):

```
For each $0 \leq k < n$
    if (  Label[$k$]= Label[$i$]  )      Label[$k$] ← Label[$j$])
```

O(n) time

Sets under operations

Given: a collection of n singleton sets $\{0\}$, $\{1\}$, $\{2\}$, … $\{n-1\}$

Aim: a compact data structure to perform

- Union(i, j):

  Unite the two sets containing i and j.

- Same_sets(i, j):

  Determine if i and j belong to the same set.

Trivial Solution 2

Treat the problem as a graph problem: ??

Connected component

- **V** = $\{0$,…, $n-1\}$, **E** = empty set initially.
- A set ⬄ a connected component.
- Keep array **Label[]** such that **Label**[i]=**Label**[j] iff i and j belong to the same component.

➔

Union(i, j) :

O(n) time

add an edge (i, j) and

recompute connected components using **BFS/DFS**.

Sets under operations

Given: a collection of $\boldsymbol{n}$ singleton sets $\{\mathbf{0}\}$, $\{\mathbf{1}\}$, $\{\mathbf{2}\}$, … $\{\boldsymbol{n}-\mathbf{1}\}$

Aim: a compact data structure to perform

- Union($\boldsymbol{i}$, $\boldsymbol{j}$):

  Unite the two sets containing $\boldsymbol{i}$ and $\boldsymbol{j}$.

- Same_sets($\boldsymbol{i}, \boldsymbol{j}$):

  Determine if $\boldsymbol{i}$ and $\boldsymbol{j}$ belong to the same set.

Efficient solution:

- A data structure which supports each operation in **O(log $\boldsymbol{n}$)** time.

- **An additional heuristic**

  ➔ time complexity of an operation : practically **O**($\mathbf{1}$).