Data Structures and Algorithms

(CS210A)

Semester I – **2014-15**

Lecture 14:

- **Algorithm paradigm of Divide and Conquer : Counting the number of Inversions**
- **Another sorting algorithm based on Divide and Conquer : Quick Sort**

Divide and Conquer paradigm
An Overview

A problem in this paradigm is solved in the following way.

1. **Divide** the problem instance into two or more instances of the same problem.
2. Solve each smaller instances **recursively** (base case suitably defined).
3. **Combine** the solutions of the smaller instances to get the solution of the original instance.

This is usually the main **nontrivial** step in the design of an algorithm using divide and conquer strategy

Role of Data Structures in designing efficient algorithms

Definition: A collection of data elements *arranged* and *connected* in a way which can facilitate efficient executions of a (possibly long) sequence of operations.

Parameters:

- Query/Update time
- Space
- Preprocessing time

Role of Data Structures in designing efficient algorithms

Definition: A collection of data elements *arranged* and *connected* in a way which can facilitate efficient executions of a (possibly long) sequence of operations.

Consider an Algorithm $\boldsymbol{A}$.

Suppose $\boldsymbol{A}$ performs many operations of same type on some data.

Improving time complexity of these operations → Improving the time complexity of $\boldsymbol{A}$.

So, it is worth designing a suitable data structure.

Counting Inversions in an array

Problem description

Definition (Inversion): Given an array **A** of size n,

a pair (i, j), $0 \le i < j < n$ is called an inversion if **A**[i]>**A**[j].

Example:

	0	1	2	3	4	5	6	7
A	3	15	8	19	9	67	11	27

Inversions are : (1,2), (1,4), (3,4), (1,6), (3,6), (5,6), (5,7)

AIM: An efficient algorithm to count the number of inversions in an array **A**.

Counting Inversions in an array

Problem familiarization

```
Trivial-algo(A[0..n − 1])
{ count ← 0;
  For(j=1  to n − 1) do
  {      For( i=0 to j − 1 )
         {      If (A[i]>A[j])  count ← count + 1;
         }
  }   return count;
}
```

Time complexity: **O**(n^2)

Question: What can be the max. no. of inversions in an array **A** ?

Answer: $\binom{n}{2}$, which is **O**(n^2).

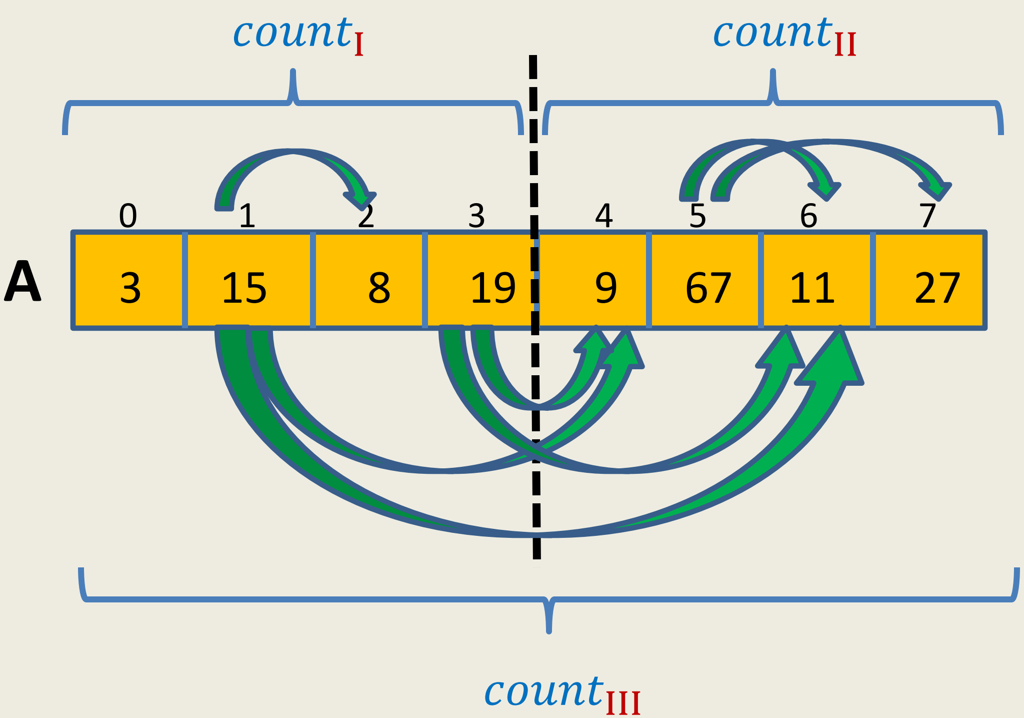
Question: Is the algorithm given above optimal ?

Answer: No, our aim is **not** to report all inversions but to report the count.

Let us try to design a

Divide and Conquer based algorithm

How do we approach using divide & conquer



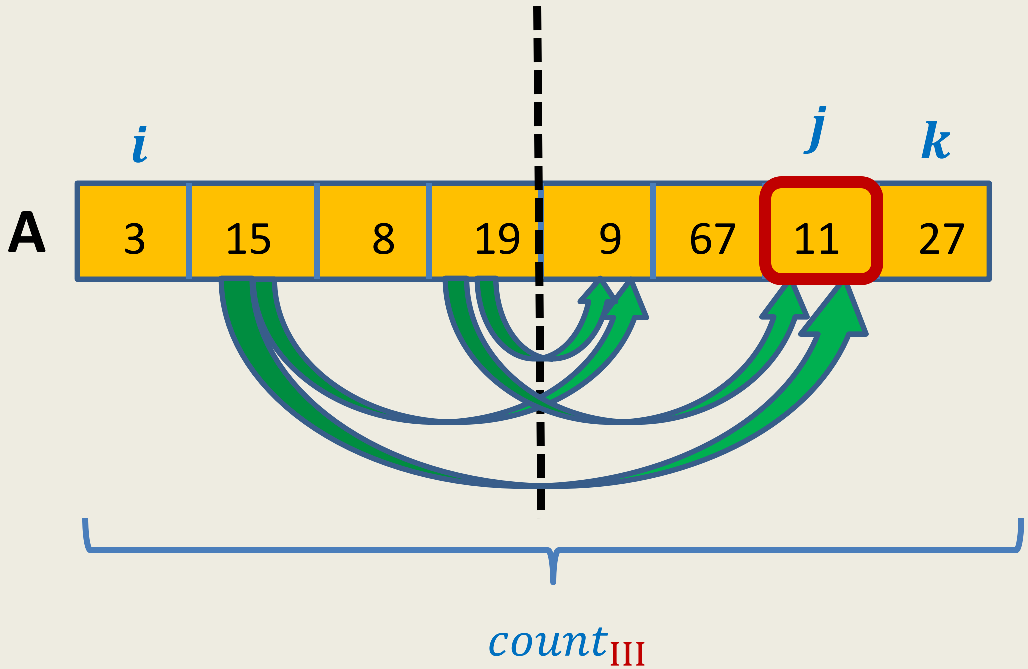
Counting Inversions
Divide and Conquer based algorithm

```
CountInversion( A,i, k)      // Counting no. of inversions in A[i..k]
If (i = k) return 0;
 Else{  mid ← (i + k)/2;
        count_I ← CountInversion(A,i, mid);
        count_II ← CountInversion(A,mid + 1, k);

          …. Code for count_III ….

      return count_I + count_II + count_III ;
    }
```

How to efficiently compute $count_{\mathrm{III}}$ (Inversions of type III) ?



$\mathbf{O}(n^2)$ time algo

Aim: For each $\mathbf{mid} < j \leq k$, count the elements in **A**[i..$\mathbf{mid}$] that are **greater** than **A**[j].

Trivial way: **O**(**size of the subarray A**[i..$\mathbf{mid}$]) time for a given j.

→ **O**(n) time for a given j in the first call of the algorithm.

→ **O**(n^2) time for computing $count_{\mathrm{III}}$ since there are $n/2$ possible values of j.

How to efficiently compute $count_{\mathrm{III}}$ (Inversions of type III) ?

Key Observation: We have to perform $\boldsymbol{n/2}$ operations of the same kind:

How many elements in **A**[$\boldsymbol{i}$..$\mathbf{mid}$] *are* **greater** *than* **A**[$\boldsymbol{j}$] ?

Lesson from Data Structures :

We should build **a suitable data structure** storing elements of **A**[$\boldsymbol{i}$..$\mathbf{mid}$] so that the above operation can be performed efficiently for any $\boldsymbol{j}$.

Question: What should be the data structure ?

Answer: Sorted subarray **A**[$\boldsymbol{i}$..$\mathbf{mid}$].

Counting Inversions

First algorithm based on divide & conquer

CountInversion(**A**,i, k)

If (i=k) return 0;

Else{ $\mathbf{mid} \leftarrow (i + k)/2$;

$count_{\mathrm{I}}$ ← **CountInversion**(**A**,i, $\mathbf{mid}$);

$count_{\mathrm{II}}$ ← **CountInversion**(**A**,$\mathbf{mid}+1$, k);

2 **T**($n/2$)

Sort(**A**,i, $\mathbf{mid}$);

For each $\mathbf{mid} < j \leq k$

do **binary search** for **A**[j] in **A**[i..$\mathbf{mid}$] to compute the *number* of elements greater than **A**[j].

Add this *number* to $count_{\mathrm{III}}$;

c n **log** n

return $count_{\mathrm{I}}$ + $count_{\mathrm{II}}$ + $count_{\mathrm{III}}$;

}

Counting Inversions
First algorithm based on divide & conquer

Time complexity analysis:

If $n = 1$,

$\mathbf{T}(n) = c$ for some constant c

If $n > 1$,

$\mathbf{T}(n) = c\, n \log n + 2\, \mathbf{T}(n/2)$

$= c\, n \log n + c\, n\, ((\log n) - 1) + 2^2\, \mathbf{T}(n/2^2)$

$= c\, n \log n + c\, n\, ((\log n) - 1) + c\, n\, ((\log n) - 2) + 2^3\, \mathbf{T}(n/2^3)$

$= \mathbf{O}(n \log^2 n)$

Can we improve it further ?

Sequence of observations
To achieve better running time

- The extra **log** n factor arises because for the "**combine**" step, we are spending **O**(n **log** n) time instead of **O**(n).

- The reason for **O**(n **log** n) time for the "**combine**" step:
  - **Sorting A**[0.. n/2] takes **O**(n **log** n) time.
  - Doing **Binary Search** for n/**2** elements from **A**[n**/2… n -1**]

- Each of the above tasks have optimal running time.

- So the only way to improve the running time of "**combine**" step is some new idea

Learn from the past knowledge

Many of you noticed some **similarity** between the code of the O($n \log^2 n$) time algorithm and **Merge Sort**. Explore these similarities more closely.

Revisiting MergeSort algorithm

MSort(**A**,i, k)// Sorting **A**[$i..k$]

{ **If** ($i < k$)

{ $\mathbf{mid} \leftarrow (i + k)/2$;

MSort(**A**,i, $\mathbf{mid}$);

MSort(**A**,$mid + 1, k$);

Create a temporary array **C**[$0..k - i$]

Merge(**A**,i, mid, k, **C**);

Copy **C**[$0..k - i$] to **A**[$i..k$]

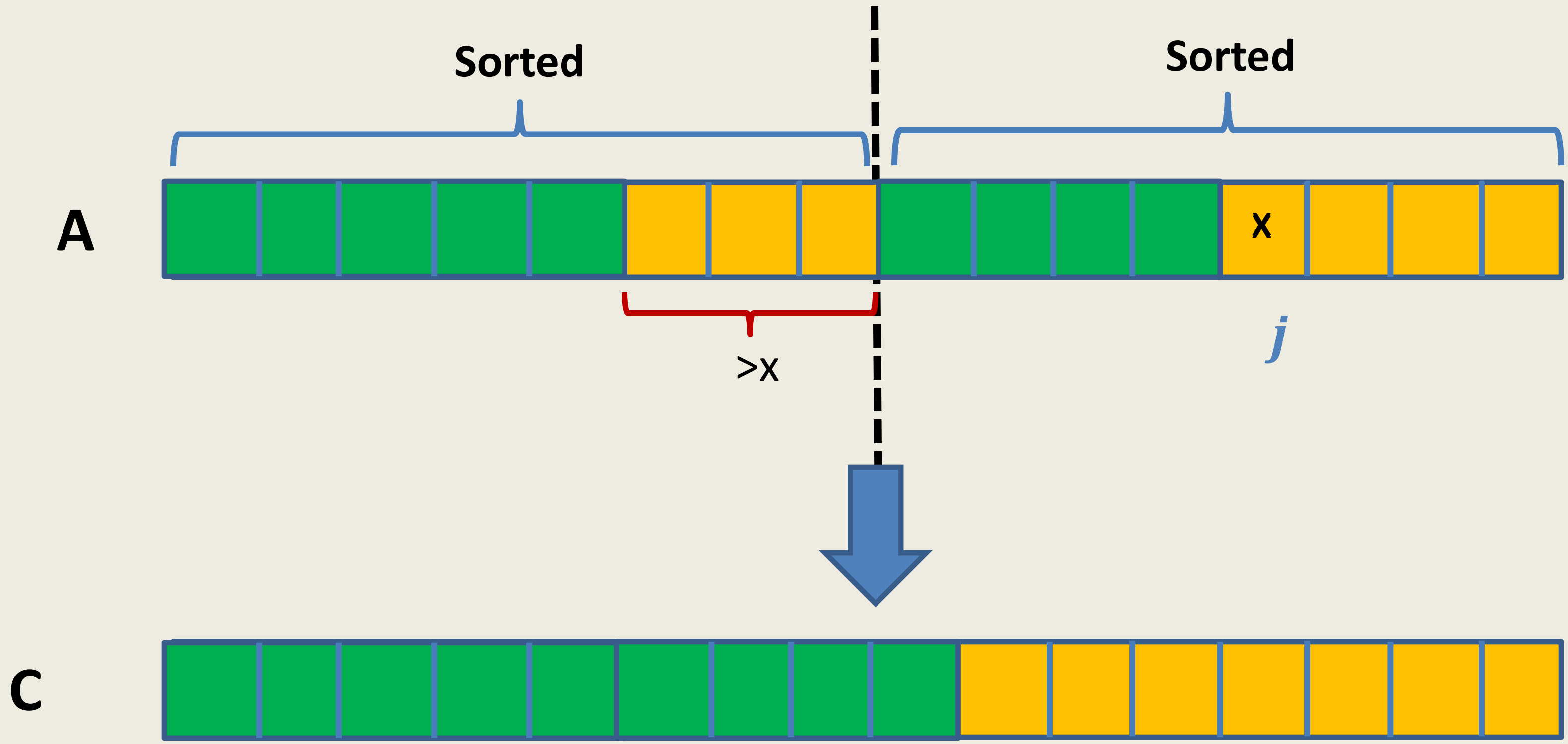
}

}

We shall carefully look at the **Merge**() procedure to find an efficient way to count the number of elements from **A**[$i..\mathbf{mid}$] which are smaller than **A**[j] for any given $\mathbf{mid} < j \leq k$

Relook

Merging A[$i..\mathbf{mid}$] and A[$\mathbf{mid}+1..k$]



Pesudo-code for Merging two sorted arrays

```
Merge(A,i, mid, k,C)
  p← i;  j← mid + 1; r← 0;
  While(p ≤ mid  and j  ≤  k)
  {       If(A[p]< A[j]) {     C[r] ← A[p]; r++;  p++  }
          Else              {     C[r] ← A[j]; r++;  j++  }
  }
  While(p ≤ mid)    {  C[k] ← A[i]; k++;  i++   }
  While(j ≤ k)         {  C[k] ← A[j]; k++;  j++   }
  return C ;
```

We shall make **just a slight change** in the above pseudo-code to achieve our main objective of computing $count_{III}$. If you understood the discussion of the previous slide, can you guess it now ?

Pesudo-code for
Merging and counting inversions

```
Merge_and_CountInversion(A,i, mid, k,C)
  p ← i;  j ← mid + 1; r ← 0;
  count_III ← 0;
  While(p ≤ mid  and j ≤ k)
  {      If(A[p]< A[j]) {    C[r] ← A[p]; r++;  p++  }
         Else            {    C[r] ← A[j]; r++;  j++
                              count_III ← count_III + (mid-p+1);
                         }
  }
  While(p ≤ mid)    { C[k] ← A[i]; k++;  i++   }
  While(j ≤ k)      { C[k] ← A[j]; k++;  j++   }
  return count_III;
```

Counting Inversions
Final algorithm based on divide & conquer

Sort_and_CountInversion(**A**, i, k)

{ **If** ($i = k$) return 0;

else

{ $\mathbf{mid} \leftarrow (i + k)/\mathbf{2}$;

$count_{\mathrm{I}}$ ← **Sort_and_CountInversion** (**A**,i, $\mathbf{mid}$);

$count_{\mathrm{II}}$ ← **Sort_and_CountInversion** (**A**,$\mathbf{mid} + \mathbf{1}, k$);

Create a temporary array **C**[$\mathbf{0}..k - i$]

$count_{\mathrm{III}}$ ← **Merge_and_CountInversion**(**A**,i, $\mathbf{mid}$, k,**C**);

Copy **C**[$\mathbf{0}..k - i$] to **A**[$i..k$];

return $count_{\mathrm{I}}$ + $count_{\mathrm{II}}$ + $count_{\mathrm{III}}$;

}

}

(first two recursive calls: 2 T($n/2$); remaining steps: O(n))

Counting Inversions
Final algorithm based on divide & conquer

Time complexity analysis:

If n = 1,

T(n) = c for some constant c

If n > 1,

T(n) = c n + **2 T**(n/**2**)

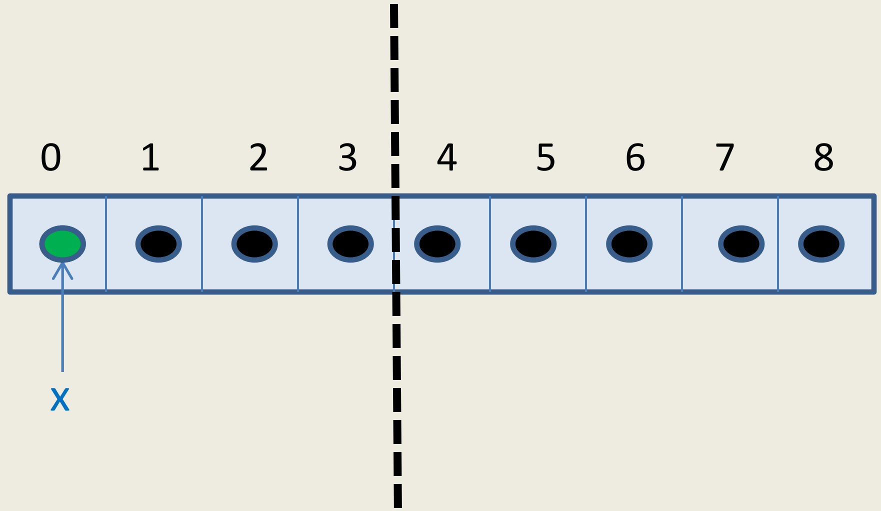
= **O**(n **log** n)

Theorem: There is **a divide and conquer** based algorithm for computing the number of inversions in an array of size n. The running time of the algorithm is **O**(n **log** n).

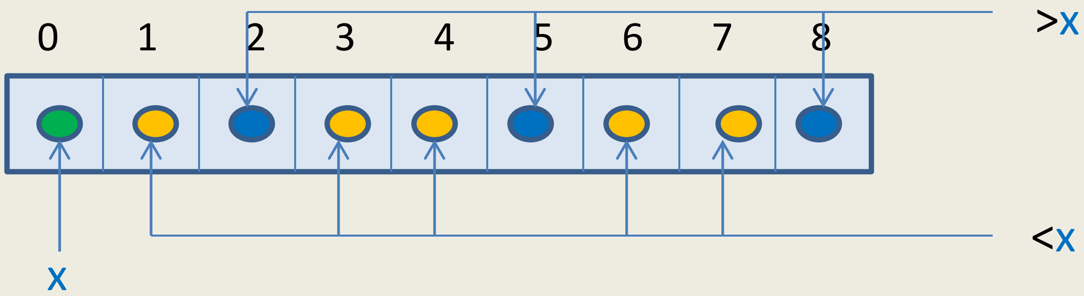
Another sorting algorithm based on divide and conquer

QuickSort

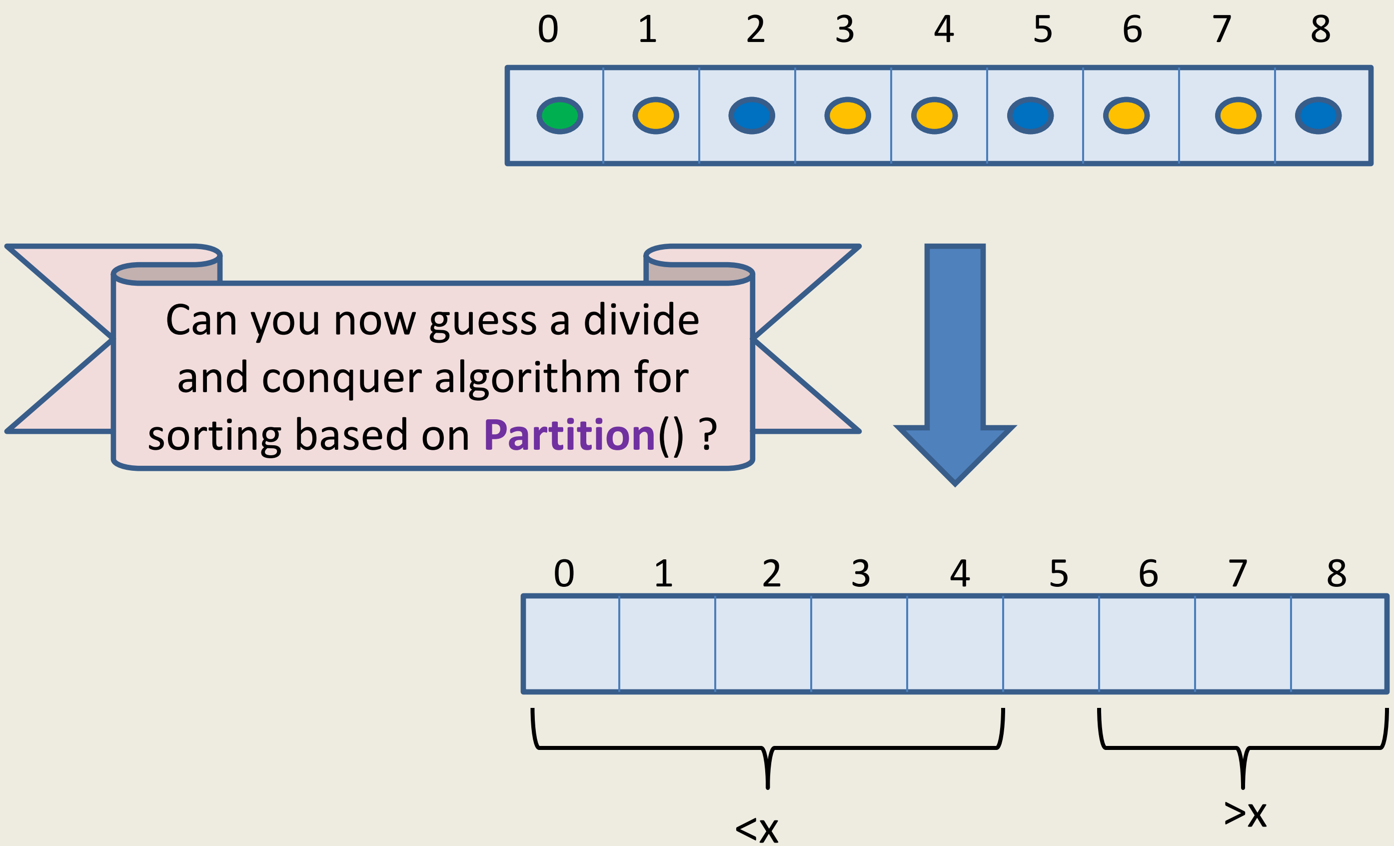
Is there any alternate way to divide ?



In MergeSort, we divide the input instance in an obvious manner.



0
1
2
3
4
5
6
7
8
>x
<x
x



This procedure is called **Partition**.

It **rearranges** the elements so that all elements less than x appear to the left of x and all elements greater than x appear to the right of x.

Pseudocode for QuickSort(S)

QuickSort(S)
{ If (|S|>1)
Pick and remove an element x from S;
($S_{<x}$, $S_{>x}$)← Partition(S,x);
return(Concatenate(QuickSort($S_{<x}$), x, QuickSort($S_{>x}$))
}

Pseudocode for QuickSort(S)

When the input S is stored in an array

QuickSort(A,l, r)

{ If (l < r)

i← Partition(A,l,r); // i is index where element $A[l]$ is finally placed

QuickSort(A,l, $i-1$);

QuickSort(A,$i+1$, r)

}

View this algorithm from various perspectives. For almost all practical purposes, this is the most efficient algorithm for sorting. It outperforms **MergeSort** by a significant factor.

QuickSort

Homework:

- The running time of Quick Sort depends upon the element we choose for partition in each recursive call. What can be the worst case running time of Quick Sort ? What can be the best case running time of Quick Sort ?

- Give an implementation of **Partition** that takes **O**($r - l$) time and using **O**(1) extra space only.

Sometime later in the course, we shall revisit **QuickSort** *and analyze its average time complexity.*