Data Structures and Algorithms

(CS210A)

Semester I – **2014-15**

Lecture 13:

- Algorithm paradigms
- Algorithm paradigm of Divide and Conquer
- Majority element : Proof of correctness of the algorithm from last class

Algorithm Paradigms

Algorithm Paradigm

Motivation:

- Many problems whose algorithms are based on a common approach.
- A need of a systematic study of the characteristics of such widely used approaches.

Algorithm Paradigms:

- **Divide and Conquer**
- **Greedy Strategy**
- **Dynamic Programming**
- **Local Search**

Divide and Conquer paradigm for Algorithm Design

Divide and Conquer paradigm
An Overview

A problem in this paradigm is solved in the following way.

1. **Divide** the problem instance into two or more instances of the same problem.
2. Solve each smaller instances **recursively** (base case suitably defined).
3. **Combine** the solutions of the smaller instances to get the solution of the original instance.

This is usually the main **nontrivial** step in the design of an algorithm using divide and conquer strategy

Example 1

Sorting

A problem in Practice sheet 1 (8 August)

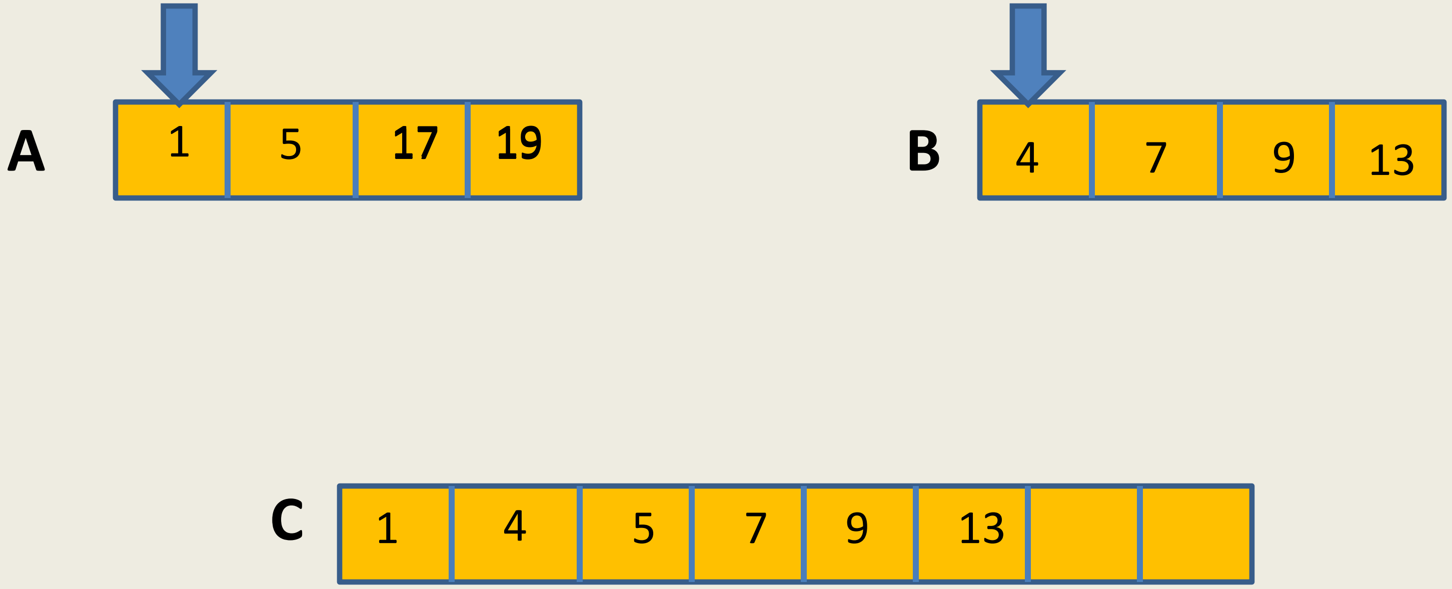
Merging two sorted arrays:

Given two sorted arrays **A** and **B** storing n elements each, Design an **O**(n) time algorithm to output a sorted array **C** containing all elements of **A** and **B**.

Example: If **A**={1,5,17,19} **B=**{4,7,9,13}, then output is

C={1,4,5,7,9,13,17,19}.

Merging two sorted arrays A and B



Pesudo-code for Merging two sorted arrays

Merge(**A**[0..n-1],**B**[0..m-1], **C**)   // Merging two sorted arrays **A** and **B** into array **C.**

Correctness : An exercise

```
{ i← 0; j← 0;
   k← 0;
  While(i<n  and j<m)
  {      If(A[i]< B[j]) {   C[k] ← A[i]; k++;  i++    }
         Else           {   C[k] ← B[j]; k++;  j++    }
  }
  While(i<n) {  C[k] ← A[i]; k++;  i++   }
  While(j<m) {  C[k] ← B[j]; k++;  j++   }
  return C;
}
```

Time Complexity = **O**(n+m)

Divide and Conquer based sorting algorithm

MSort(**A**,i,j) // Sorting the subarray **A**[$i..j$].

{ **If** ($i < j$)

{ **mid** ← (i+j)/2;

MSort(**A**,i,**mid**);

MSort(**A**,**mid**+1,j);

} Divide step

Create temporarily **C**[$0..j-i$]

Merge(**A**[i..**mid**], **A**[**mid**+1..j], **C**);

Copy **C**[$0..j-i$] to **A**[$i..j$]

} Combine/conquer step

}

This is Merge Sort algorithm

Divide and Conquer based sorting algorithm

```
MSort(A,i,j)  // Sorting the subarray A[i..j].
{  If ( i< j )
   {  mid ← (i+j)/2;
      MSort(A,i,mid);              ← T(n/2)
      MSort(A,mid+1,j);            ← T(n/2)
      Create temporarily C[0..j − i]          ┐
      Merge(A[i..mid], A[mid+1..j], C);       ├ c n
      Copy C[0..j − i] to A[i..j]             ┘
}
```

Time complexity:

If $n = 1$,

$T(n) = c$ for some constant c

If $n > 1$,

$T(n) = c\,n + 2\,T(n/2)$

$= c\,n + c\,n + 2^2\,T(n/2^2)$

$= c\,n + c\,n + c\,n + 2^3\,T(n/2^3)$

$= c\,n + \ldots(\log n \text{ terms})\ldots + c\,n$

$= O(n \log n)$

Proof of correctness of Merge-Sort

MSort(A, i**,** j**)** // Sorting the subarray **A**[$i..j$].

{ **If** (i< j)

{ **mid** ← (i+j)/2;

MSort(A, i**,mid)**;

MSort(A,mid+1,j);

Create temporarily **C**[$0..j-i$]

Merge(**A**[i..**mid**], **A**[**mid**+1..j]**, C**);

Copy **C**[$0..j-i$] to **A**[$i..j$]

}

Question: What is to be proved ?

Answer: **MSort(A,**i**,**j**)** sorts the subarray **A**[$i..j$]

Question: How to prove ?

Answer:

- By **induction** on the length ($j-i+1$) of the subarray.
- Use correctness of the algorithm **Merge.**

Example 2

Faster algorithm for multiplying two integers

Addition is faster than multiplication

Given: any two n-bit numbers **X** and **Y**

Question: how many **bit-operations** are required to compute **X+Y** ?

Answer: **O**(n)

Question: how many **bit-operations** are required to compute **X*** $\mathbf{2^n}$?

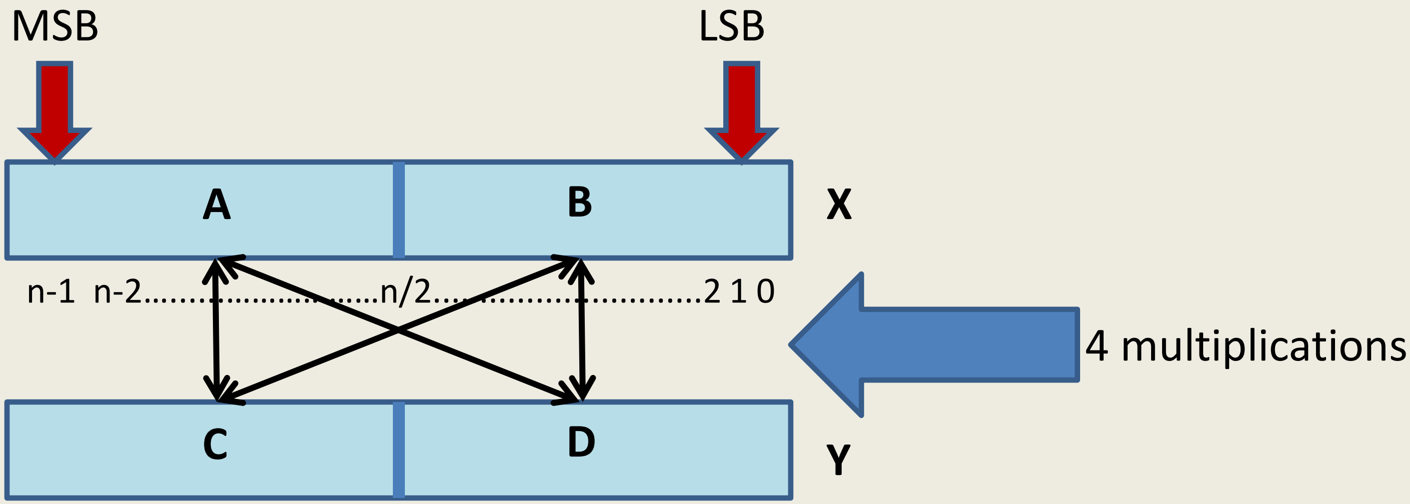
Answer: **O**(n) [left shift the number **X** by n places, (do it carefully)]

Question: how many **bit-operations** are required to compute **X*Y** ?

Answer: **O**(n^2) (why ??)

Can we compute X*Y faster ??

Pursuing Divide and Conquer approach



Question: how to express **X*Y** in terms of **multiplication/addition** of **{A,B,C,D} ?**

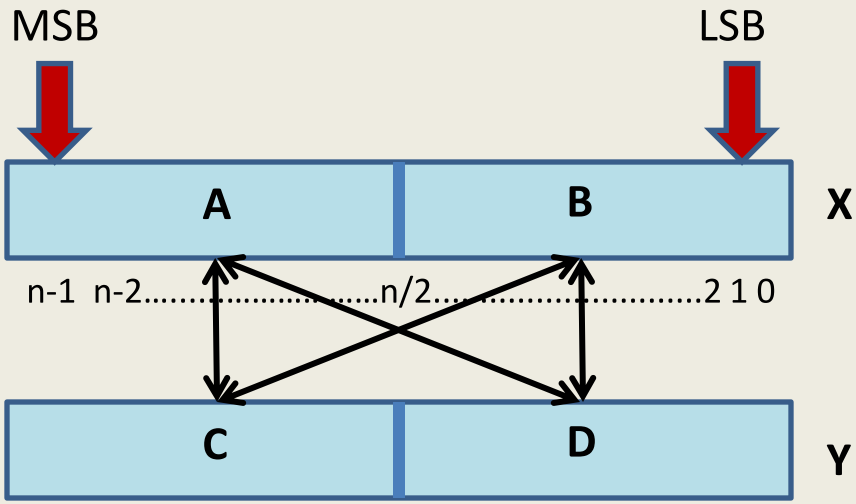
Hint: First Express **X** and **Y** in terms of **{A,B,C,D}.**

$$X = A * 2^{n/2} + B \qquad \text{and } Y = C * 2^{n/2} + D$$

Hence …

$$X * Y = (A * C) * 2^{n} + (A * D + B * C) * 2^{n/2} + B * D$$

Pursuing Divide and Conquer approach



$$X*Y = (A*C)* \boxed{2^n} + (A*D + B*C)* \boxed{2^{n/2}} + B*D$$

Let $T(n)$: time complexity of multiplying **X** and **Y** using the above equation.

$T(n) = c\,n + 4\,T(n/2)$ for some constant c

$= c\,n + 2c\,n + 4^2\,T(n/2^2)$

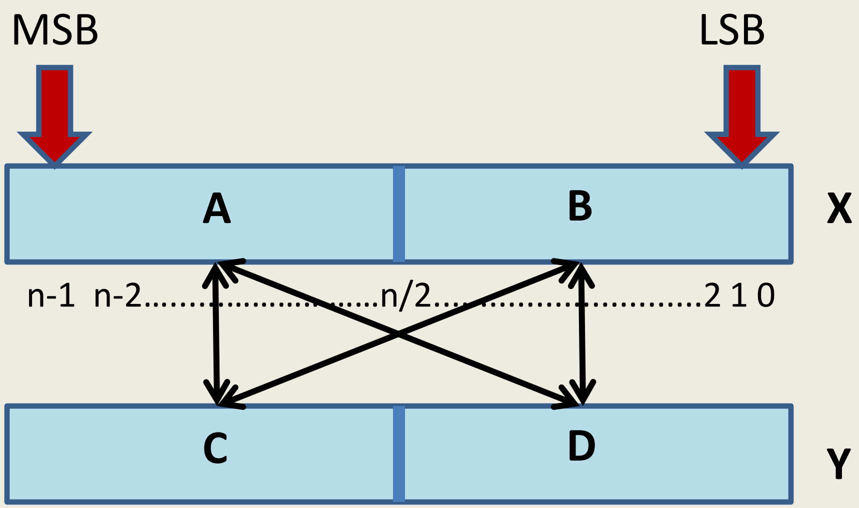
$= c\,n + 2c\,n + 4c\,n + 4^3\,T(n/2^3)$

$= c\,n + 2c\,n + 4c\,n + 8c\,n + \ldots + 4^{\log_2 n}T(1)$

$= c\,n + 2c\,n + 4c\,n + 8c\,n + \ldots + c\,n^2$

$O(n^2)$ time algo

Pursuing Divide and Conquer approach



X*Y = (A*C)* 2^n + (A*D + B*C)* $2^{n/2}$ + B*D

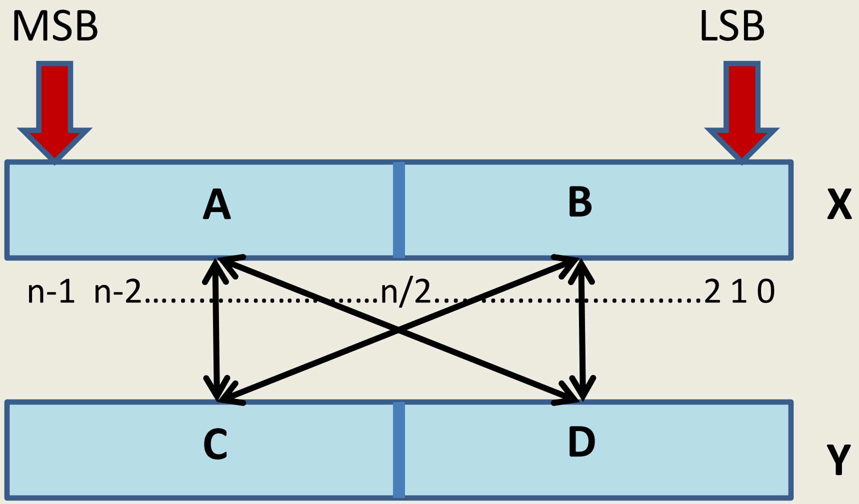
Observation: A*D + B*C = (A-B)*(D-C) + A*C + B*D

Question: How many multiplications do we need **now** to compute **X*Y** ?

Answer: 3 multiplications :

- **A*C**
- **B*D**
- **(A-B)*(D-C) .**

Pursuing Divide and Conquer approach



$$X*Y = (A*C)* 2^n + ((A-B)*(D-C) + A*C + B*D) 2^{n/2} + B*D$$

Let $T(n)$: time complexity of the **new algo** for multiplying two n-bit numbers

$T(n) = c\,n + 3\,T(n/2)$ for some constant c

$= c\,n + 3\,c\frac{n}{2} + 3^2\,T(n/2^2)$

$= c\,n + 3c\frac{n}{2} + 9c\frac{n}{4} + + \ldots + 3^{\log_2 n}\,T(1)$

$= O(n^{\log_2 3}) = O(n^{1.58})$

Conclusion

Theorem: There is a **divide and conquer** based algorithm for multiplying any two n-bit numbers in **O($n^{1.58}$)** time **(bit operations).**

Note:

The fastest algorithm for this problem runs in almost **O(n log n)** time.

One such algorithm was designed in **2008** at CSE, IIT Kanpur.

By (Dey, Kurur, Saha, and Saptharishi).

Example 3

Counting the number of "*inversions*" in an array

Counting Inversions in an array

Problem description

Definition (Inversion): Given an array **A** of size n, a pair (i,j), $0 \le i < j < n$ is called an inversion if **A**[i]>**A**[j].

Example:

	0	1	2	3	4	5	6	7
A	3	15	8	19	9	67	11	27

Inversions are : (1,2), (1,4), (3,4), (1,6), (3,6), (5,6), (5,7)

AIM: An efficient algorithm to count the number of inversions in an array **A**.

Counting Inversions in an array
Problem familiarization

```
Trivial-algo(A[0..n-1])
{ count ← 0;
  For(j=1  to n-1) do
  {      For( i=0 to j-1 )
         {      If (A[i]>A[j])  count ← count + 1;
         }
  }
}
```

Ponder over the **divide and conquer algorithm** for this problem. We shall discuss it in the next class.

Time complexity: O(n^2)

Question: What can be the max. no. of inversions in an array **A** ?

Answer: $\binom{n}{2}$, which is O(n^2).

Question: Is the algorithm given above optimal ?

Answer: No, our aim is **not** to report all inversions but to report the count.

Proof of correctness

Algorithm for majority element

Algorithm for majority element

```
Algo-majority(A){
    count← 0;
    for(i=0 to n-1)
    {     if (count=0)
          {   x← A[i];
              count← 1;
          }
           else if(x<> A[i])
                  count ← count - 1;
                  else
                   count← count + 1;
    }
  if x appears  more than n/2, then
       print(x is majority element)
  else print(no majority element)
```

Question: What is to be proved ?

Answer: For every possible instance of A, the output of algorithm is correct.

Observation: If A does not have any majority element, the output of the algorithm is correct.

➔

Inference: To prove correctness, it suffices to prove the following:

> If A has a majority element, say α, then at the end of the **For** loop, x=α

Algorithm for majority element

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    count← 0;
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          {   x← A[i];
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          else if(x<> A[i])
                count ← count - 1;
                else
                count← count + 1;
    }
   if x appears  more than n/2, then
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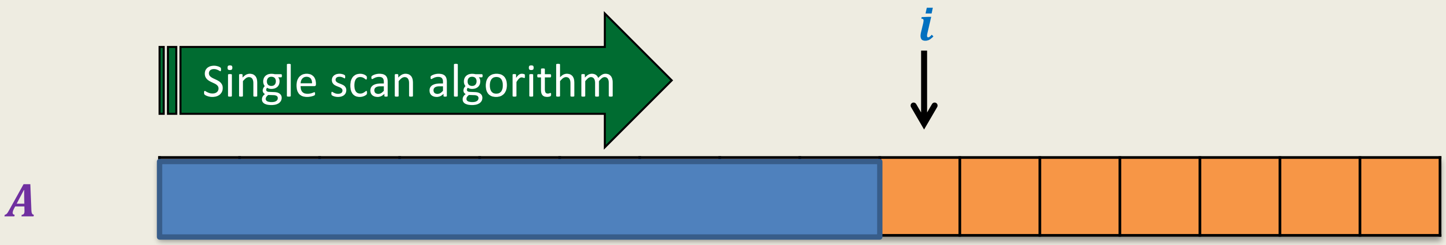
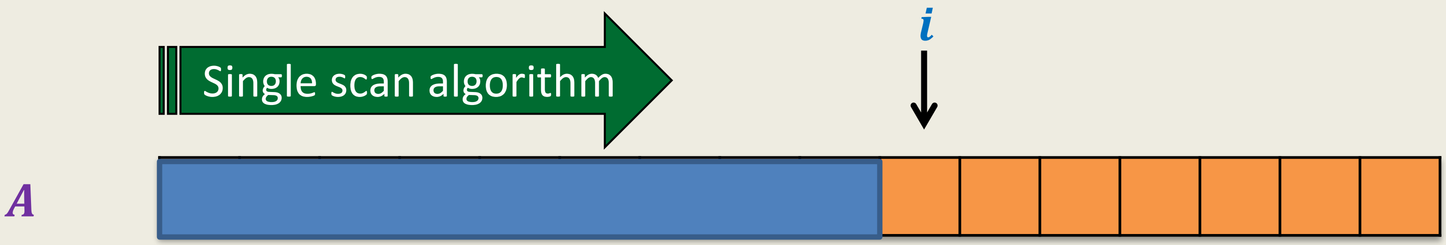
Inference: To prove correctness, it suffices to prove the following:

> If **A** has a majority element, say **α**, then at the end of the **For** loop, **x**=**α**



Question: What assertion holds at the end of ith iteration ?

A wrong answer: **x** is a majority element of {**A**[0],**A**[1],…,**A**[i-1]} ?



Question: What assertion holds at the end of ith iteration ?

Answer:

P(i) : **α** is a majority element of {**x**,…count times…,**x**, **A**[i],…,**A**[n-1]} ?

Question: What is **P(n) ?**

Answer: **α** is a majority element of {**x**,…count times…,**x**}

➔ **x = α**

As a homework exercise, prove assertion **P(i)** by induction on i.