Irreducible and Prime Ideals

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Let us start with definitions.

Definition 0.1. Let R be a commutative ring. Ideal I of R is *irreducible* if whenever $I = I_1 \cdot I_2$ for ideals I_1 and I_2, either $I_1 = (1)$ or $I_2 = (1)$. Ideal I is *prime* if whenever $a \cdot b \in I$ for $a, b \in R$, either $a \in I$ or $b \in I$.

These two definitions are incomparable in general. We show this by two examples.

Prime but not irreducible. Let $R = F[x]/(x^2 - x)$ for a field F. Elements of R are of the form $a + bx$, $a, b \in F$. Let $I = (x)$. Ideal I is not irreducible since $(x) \cdot (x) = (x^2) = (x)$. On the other hand, if $(a + bx) \cdot (c + dx) \in I$, we have

$$ac + (bc + ad) \cdot x + bd \cdot x^2 = ac + (bc + ad + bd)x \in I.$$

This gives $ac \in I$ and hence either $a = 0$ or $c = 0$, proving that I is prime.

Irreducible but not prime. Let $R = Z_4[x]$ and $I = (x)$. Let $I = I_1 \cdot I_2$. Both I_1 and I_2 contain the ideal I. If $I_1 = I_2 = I$, then $I_1 \cdot I_2 = (x^2) \neq I$. Hence, at least one of I_1 or I_2 is larger than I. Let it be I_1. Then I_1 contains a for $a \in Z_4$. If $a = 1$ or 3, then $I_1 = (1)$. Suppose $a = 2$. Then $I_1 = (2, x)$. Ideal I_2 either equals I, or $(2, x)$, or (1) by a similar argument. If $I_2 = (x)$ or $(2, x)$, then $I_1 \cdot I_2 = (2x, x^2) \neq I$. Hence we must have either $I_1 = (1)$ or $I_2 = (1)$.

Ideal I is not prime since $2 \cdot 2 = 0 \in I$, but $2 \notin I$.

In contrast, for Dedekind domains, both the definitions coincide.

Theorem 0.2. *If R is a Dedekind domain, then I is irreducible iff I is prime.*

Proof. Suppose I is prime and $I = I_1 \cdot I_2$. In a Dedekind domain, every ideal can be uniquely written as a product of prime ideals. Since I is prime, the only way to write I as a product is $I \cdot (1)$. Hence $I_1 = (1)$ or $I_2 = (1)$.

Suppose I is not prime, then $I = I_1 \cdot I_2 \cdot \cdots \cdot I_k$ with each I_k a prime ideal and $k > 1$. Hence I is not irreducible. $\square$