Completions of $\mathbb{Q}$

August 29, 2016

1 From $\mathbb{Q}$ to $\mathbb{R}$

The notion of numbers started from natural numbers (1, 2, 3, 4, ...). In order to make the definitions of addition and subtraction robust, the number zero and negative numbers were added to it to arrive at the notion of integers ($\mathbb{Z}$). Rational numbers ($\mathbb{Q}$) followed naturally to make division robust. The next step was a bit more problematic: reals. Initially, it was believed that all the numbers are rationals, but many suspected that there are more numbers. It was confirmed when $\sqrt{2}$ was proven to be an irrational number. This forced everyone to change their views as $\sqrt{2}$ can easily be realized as the length of hypotenuse of a right angle triangle with other two sides of length 1. The question then arose: how does one define the set of real numbers? To use the standard definition that is taught in school is not very precise: the collection of all numbers that have possibly infinite decimal expansion. In this definition, there are more than one representations of a number. For example, $2.000000\cdots$ is the same as $1.999999\cdots$. Even worse, it is not clear how to add or multiply two real numbers in this representation! This is because for a general real number, its decimal expansion will continue infinitely and without any pattern. Hence given two real numbers, their addition cannot be defined in the same way as for rational numbers.

Thus we have a serious problem at our hand. While we know intuitively that irrational real numbers can be added and multiplied, we are unable to define it precisely. These problems were resolved by mathematician Cauchy through the idea of reals being limits of convegring sequence of rationals. *Rings played a key role*!

Definition 1.1. Let $(a_0, a_1, a_2, \ldots)$ be an infinite sequence of rational numbers. We say that such a sequence is a *Cauchy sequence* if for every rational number $\epsilon > 0$, there exists an $m > 0$ such that for all $n \geq m$: $|a_n - a_m| < \epsilon$.

A Cauchy sequence has very strong convergence property: beyond a point, any two numbers in the sequence must be very close to each other.

Definition 1.2. A Cauchy sequence $s = (a_0, a_1, \ldots)$ *converges* to a rational number c if for every rational $\epsilon > 0$, there exists an $m > 0$ such that for every $n \geq m$: $|c - a_n| < \epsilon$.

For example, the sequence $(1, 1, 1, 1, 1, \ldots)$ converges to 1. The sequence $(1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots, 1 + \frac{1}{m}, \ldots)$ also converges to 1. There are Cauchy sequences that do not converge to a rational number, and showing this can be non-trivial. For example:

Lemma 1.3. *The sequence* $(1, 1 + \frac{1}{2}, 1 + \frac{1}{2 + \frac{1}{2}}, 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}, \ldots)$ *is Cauchy and converges to* $\sqrt{2}$.

Proof. Let $1 + \frac{p_n}{q_n}$ be the nth term of the sequence with $p_n, q_n \in \mathbb{Z}$. We have $p_0 = 1$, $q_0 = 2$, and $\frac{p_n}{q_n} = \frac{1}{1+\frac{p_{n-1}}{q_{n-1}}}$ for $n \geq 1$. Therefore,

$$\frac{p_n}{q_n} = \frac{q_{n-1}}{p_{n-1} + q_{n-1}}.$$

We also have that p_n and q_n are relatively prime (prove this inductively: it is true for $n = 0$, and if true for $n-1$, must be true for n). This gives: $p_n = q_{n-1}$ and $q_n = p_{n-1} + q_{n-1}$.

The difference between two consecutive terms is:

$$\begin{aligned}
\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} &= \frac{p_n q_{n-1} - q_n p_{n-1}}{q_n q_{n-1}} \\
&= \frac{q_{n-1}(p_{n-2} + q_{n-2}) - (p_{n-1} + q_{n-1})q_{n-2}}{q_n q_{n-1}} \\
&= \frac{q_{n-1}p_{n-2} - p_{n-1}q_{n-2}}{q_n q_{n-1}} \\
&= \cdots \\
&= (-1)^n \frac{p_1 q_0 - q_1 p_0}{q_n q_{n-1}} \\
&= (-1)^n \frac{1}{q_n q_{n-1}}.
\end{aligned}$$

Recall that $q_n = q_{n-1} + p_{n-1} = q_{n-1} + q_{n-2} \geq 2q_{n-2}$. Hence,

$$\left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right| \leq \frac{1}{2^{n-2}}.$$

So the difference between mth and nth term is:

$$\begin{aligned}
\left| \frac{p_n}{q_n} - \frac{p_m}{q_m} \right| &= \left| \sum_{\ell=m+1}^{n} \frac{p_\ell}{q_\ell} - \frac{p_{\ell-1}}{q_{\ell-1}} \right| \\
&= \left| \sum_{\ell=m+1}^{n} (-1)^\ell \frac{1}{q_\ell q_{\ell-1}} \right| \\
&\leq \sum_{\ell=m+1}^{n} \frac{1}{q_\ell q_{\ell-1}} \\
&\leq \sum_{\ell=m+1}^{n} \frac{1}{2^{\ell-2}} \\
&\leq \frac{1}{2^{m-2}}.
\end{aligned}$$

Therefore, the sequence is Cauchy. Let x be the number it converges to. Then,

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}.$$

In other words,

$$x = 1 + \frac{1}{1+x}.$$

Simplifying, we get:

$$x^2 + x = 2 + x$$

and therefore, $x = \sqrt{2}$. □

In fact, every real number corresponds to a Cauchy sequence:

Lemma 1.4. *Let $a = m.d_1 d_2 d_3 \cdots$ with d_j between 0 and 9 for $j > 0$ be the usual representation of a real number. Then the sequence $(m, m + \frac{d_1}{10}, m + \frac{d_1}{10} + \frac{d_2}{100}, \ldots)$ is Cauchy and converges to s.*

Proof. The sequence is Cauchy since the difference between its mth and nth term is

$$\sum_{\ell=m+1}^{n} \frac{d_\ell}{10^\ell} < \sum_{\ell=m+1}^{n} \frac{10}{10^\ell} < \frac{1}{10^{m-1}}.$$

It converges to the number a because the difference between a and mth term of the sequence is also bounded by $\frac{1}{10^{m-1}}$. □

It is also clear that any Cauchy sequence converges to a real number. This way of defining real numbers, however, appears even worst than the school way as for every real number, there are infinitely many Cauchy sequences converging to it! For example, sequence $(1, \frac{1}{k}, \frac{1}{2k}, \frac{1}{3k}, \ldots)$ converges to 0 for any value of k. So we have not yet solved the problem of finding a unique representation of real numbers. However, we can easily add and multiply Cauchy sequences as component wise addition and multiplication of individual rational numbers in the sequence. To see that this works as required, let

$$R = \{s \mid s \text{ is a Cauchy sequence}\},$$

with addition and multiplication of sequences defined as above. With this definition, we have:

Theorem 1.5. *R is a commutative ring.*

Proof. For a Cauchy sequence $s = (a_0, a_1, \ldots)$, denote $m_s(\epsilon)$ to be the smallest number m for which $|a_n - a_m| \leq \epsilon$. Let $s_1 = (a_0, a_1, a_2, \ldots)$, and $s_2 = (b_0, b_1, b_2, \ldots)$ be two Cauchy sequences. Then

$$s_1 + s_2 = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$$

is a Cauchy sequence since for any $\epsilon > 0$,

$$|a_n + b_n - a_m - b_m| \leq |a_n - a_m| + |b_n - b_m| \leq \epsilon,$$

where $m = \max\{m_{s_1}(\frac{\epsilon}{2}), m_{s_2}(\frac{\epsilon}{2})\}$.

Similarly,

$$s_1 * s_2 = (a_0 * b_0, a_1 * b_1, a_2 * b_2, \ldots)$$

is a Cauchy sequence since for every $\epsilon > 0$,

$$|a_n * b_n - a_m * b_m| = |a_n * (b_n - b_m) + b_m * (a_n - a_m)| \leq |a_n| \frac{\epsilon}{2\,|a_n|} + |b_m| \frac{\epsilon}{2\,|b_m|} = \epsilon,$$

where $m = \max\{m_{s_1}(\frac{\epsilon}{2A}), m_{s_2}(\frac{\epsilon}{2B})\}$ and A is the smallest positive integer greater than absolute value of every number in s_1, and B is the smallest positive integer greater than absolute value of every number in s_2.

Additive identity is the sequence $(0, 0, 0, \ldots)$ and multiplicative identity is the sequence $(1, 1, 1, \ldots)$. Additive inverse of sequence $s = (a_0, a_1, \ldots)$ is $(-a_0, -a_1, \ldots)$. □

So the ring R does capture arithmetic over reals. However, it is not a field as division by any sequence converging to zero is not possible. We fix this in the following way. Let us collect all Cauchy sequences converging to number 0:

$$I = \{s \mid \ s \text{ is a Cauchy sequence converging to } 0\}.$$

As the naming suggests:

Theorem 1.6. *I is an ideal of R.*

Proof. Let $s_1, s_2 \in I$ with $s_1 = (a_0, a_1, a_2, \ldots)$ and $s_2 = (b_0, b_1, b_2, \ldots)$. By the definition of convergence, for every rational $\epsilon > 0$, there exists m_1 and m_2 such $|\, a_n\,| < \frac{\epsilon}{2}$ and $|\, b_n\,| < \frac{\epsilon}{2}$ for every $n \geq m = \max\{m_1, m_2\}$. Therefore, for every $n \geq m$, $|\, a_n + b_n\,| < \epsilon$ and so $s_1 + s_2 \in I$. It is clear that the additive identity (all zero sequence) is in I and so is negation of any sequence in I. Therefore, I is a commutative group under addition.

Now consider $s = (a_0, a_1, a_2, \ldots) \in I$ and $t = (b_0, b_1, b_2, \ldots) \in R$. For every rational $\epsilon > 0$, there exists m such that for every $n \geq m$, $|\, a_n\,| < \frac{\epsilon}{B}$ where B is the smallest integer greater than every number in t. Hence, $|\, s_n * t_n\,| < \frac{\epsilon}{B} * \,|\, t_n\,| \leq \epsilon$. So I is closed under multiplication by any element of R and therefore is an ideal of R. □

Ideal I is in fact a *maximal ideal*:

Lemma 1.7. *I is a maximal ideal of R.*

Proof. Suppose there exists an ideal J of R such that $I \subset J$. Consider a sequence $s \in J$ that is not in I. Let $s = (a_0, a_1, a_2, \ldots)$. Since s is Cauchy, for every rational $\epsilon > 0$ and for every $n, m \geq m_s(\epsilon)$, $|\, a_n - a_m\,| \leq |\, a_n - a_{m_s(\epsilon)}\,| + |\, a_m - a_{m_s(\epsilon)}\,| < \epsilon + \epsilon = 2\epsilon$.

Moreover, since $s \notin I$, there exists a rational $\delta > 0$ such that for every $\ell > 0$, there exists an $k \geq \ell$ with $|\, a_k\,| \geq \delta$ (this is directly from the definition of convergence to a number). Choose $\epsilon = \frac{\delta}{4}$ and $\ell = m_s(\epsilon)$. Then, for every $n \geq \ell$:

$$\delta \leq |\, a_k\,| = |\, a_k - a_n + a_n\,| \leq |\, a_k - a_n\,| + |\, a_n\,| \leq 2\epsilon + |\, a_n\,| = \frac{\delta}{2} + |\, a_n\,| .$$

Hence, in the sequence s, every number beyond a_ℓ is at least $\frac{\delta}{4}$ in absolute value.

Let us modify s a bit:

$$\hat{s} = (\underbrace{1, 1, 1, \ldots, 1}_{\ell \text{ times}}, a_{\ell+1}, a_{\ell+2}, \ldots).$$

Clearly, $\hat{s} \in R$. The sequence

$$s - \hat{s} = (a_0 - 1, a_1 - 1, a_2 - 1, \ldots, a_\ell - 1, 0, 0, \ldots),$$

is in I as it converges to 0. Since $s \in J$, $\hat{s} \in J$. Also, $\hat{s} \notin I$ since $s \notin I$.

Define a new sequence t as:

$$t = \left(\underbrace{1, 1, 1, \ldots, 1}_{\ell \text{ times}}, \frac{1}{a_{\ell+1}}, \frac{1}{a_{\ell+2}}, \ldots \right).$$

Sequence t is Cauchy:

$$| \frac{1}{a_m} - \frac{1}{a_n} | = | \frac{a_n - a_m}{a_n a_m} | \leq \frac{16 \ | a_n - a_m |}{\delta^2},$$

for any $n, m > \ell$. As s is Cauchy, the difference $| a_n - a_m |$ gets smaller and smaller as n and m increase and so the same is true for t. The product

$$\hat{s} * t = (1, 1, 1, \ldots)$$

showing that $(1, 1, 1, \ldots) \in J$ and so $J = R$. Hence I is maximal. □

The quotient ring R/I clearly a field. Its elements are equivalence classes of the form $s + I$. The following propery of these equivalence classes is all we need:

Theorem 1.8. *Two sequences of R converge to same number if and only if they belong to the same equivalence class of R/I.*

Proof. Let s and t be two sequences converging to the same number c. Then $s - t$ converges to 0, and so $s - t \in I$ showing s and t to be in the same equivalence class. Conversely, if $s \in t + I$ then s converges to the same number as t since every sequence in I converges to 0. □

So $\mathbb{R} = R/I$!

The process of starting with a field F, considering Cauchy sequences of elements of the field, and then quotienting with an approproate maximal ideal is called *completion of F*. It can be shown that completion of $\mathbb{R}$ is also $\mathbb{R}$, and so we do not get anything bigger. To define Cauchy sequences for a field F, we need the notion of *valuation* defined over F that measures closeness of two elements of the field, which in our case was $| \cdot |$. Are there other ways of defining a valuation? If yes, can we define them for $\mathbb{Q}$? If yes, do we get a different completion than $\mathbb{R}$? We provide answer to all these questions in the next section.