CS203B: Final Examination

September 22, 2017

Duration: 2 Hours | Maximum Marks: 50

Modules are algebraic structures that generalize both commutative groups and rings. Their definition is the following. Let R be a ring. Structure $(M, +, \cdot)$ is an *R-module* if M is a commutative group under $+$, and $\cdot : R \times M \mapsto M$ such that for every r, $s \in R$ and a, $b \in M$:

- $(r + s) \cdot a = r \cdot a + s \cdot a$,
- $r \cdot (a + b) = r \cdot a + r \cdot b$, $(\lambda a).b = \lambda.(a.b)$
- $(rs) \cdot a = r \cdot (s \cdot a)$, and $\lambda.a$
- $1 \cdot a = a$.

Question 1. (10 marks) Prove that $(G, +)$ is a commutative group if and only if $(G, +, \cdot)$ is a $\mathbb{Z}$-module. Here, operation $\cdot$ is defined as: $n \cdot a = \underbrace{a + a + \cdots + a}_{n \text{ times}}$ for $n \in \mathbb{Z}$ and $a \in G$.

Question 2. (10 marks) Let R be a commutative ring. Let $I \subseteq R$. Prove that I is an ideal of R if and only if it is an R-module.

Question 3. (5 + 5 marks) Modules allow us to define the notion of a *fractional ideal* in integral domains. Let R be an integral domain and F be its *field of fractions* defined as:

$$F = \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\}.$$

Prove that F is a field. Let I be an ideal of R. Define

$$J = \{\alpha \mid \alpha \in F, \text{ and for all } a \in I\text{: } \alpha a \in R\}.$$

Prove that J is an R-module. J is also called a *fractional ideal.*

Question 4. (5 + 10 + 5 marks) When R is a Dedekind domain, then fractional ideal J above acts as *inverse* of I: $J \cdot I = (1)$! Let us see an example of this. We know that $\mathbb{Z}[i\sqrt{5}]$ is a Dedekind domain. Define

$$\mathbb{Q}[i\sqrt{5}] = \{\alpha + i\beta\sqrt{5} \mid \alpha, \beta \in \mathbb{Q}\}.$$

First show that $\mathbb{Q}[i\sqrt{5}]$ is the field of fractions of $\mathbb{Z}[i\sqrt{5}]$. Next, consider the ideal $I = (3, 2 + i\sqrt{5})$ of $\mathbb{Z}[i\sqrt{5}]$. Show that the fractional ideal J corresponding to I is generated by $\frac{1+\sqrt{-5}}{3}$ and 1. Finally, show that $J \cdot I = (1)$.