CS202A Endsem Exam Solutions

Q1 **(a)** We know that $a \to b \equiv \neg(a \wedge \neg b)$

$$\Rightarrow (a \wedge \neg b) \equiv \neg(a \to b)$$
$$\Rightarrow a \wedge b \equiv \neg(a \to \neg b)$$

(b) Consider any proposition ϕ constructed using logical connectives $\vee$ and $\wedge$ only. It is easy to see (formally by induction on the length of ϕ) that for valuation where all atomic propositions are assigned truth value **false**, ϕ evaluates to **false**. So ϕ can't be a tautology.

(c) Rule $\vee i1$ becomes $\dfrac{A}{\neg(\neg A \wedge \neg B)}$. We give a ND derivation of it below.

1. A Premise

——-Box B1 opens——

2. $\neg A \wedge \neg B$ Assumption
3. $\neg A$ $\wedge e$ 2
4. $\perp$ $\neg e$ 1,3

——-Box B1 closes——

5. $\neg(\neg A \wedge \neg B)$ $\neg i$ 2-4

- Proof for rule $\vee i2$ is almost the same.

For $\vee e$ we need to show that given derivations $A \vdash \theta$ and $B \vdash \theta$ in the new system, we can derive $\neg(\neg A \wedge \neg B) \vdash \theta$.

We give a ND derivation of this below.

The derivations $A \vdash \theta$, $B \vdash \theta$ are used from 3-(3.s) and 6-(6.t) in the proof below.

1. $\neg(\neg A \wedge \neg B)$
———-Box B1 opens———
2. $\neg\theta$ Assumption
———-Box B2 opens———
3. A Assumption
****
(3.1)
...
(3.s) θ
****
4. $\perp$ $\neg e$ 2, (3.s)
———-Box B2 closes———
5. $\neg A$ $\neg i$ 3-4
———-Box B2 opens———
6. B Assumption
****
(6.1)
...
(6.t) θ
****
7. $\perp$ $\neg e$ 2, (6.t)
———–Box B2 closes———
8. $\neg B$ $\neg i$ 6-7
9. $\neg A \wedge \neg B$ $\wedge i$ 5,8
10. $\perp$ $\neg e$ 1,9
———-Box B1 closes———
11. θ PBC 2-10

Q2 (a)

1. $\neg\forall x\phi$ Premise

   ———-Box B1 begins———

2. $\neg\exists x\neg\phi$ Assumption

   ———-Box B2 begins———

   x_0

   ———-Box B3 begins———

3. $\neg\phi[x_0/x]$ Assumption

4. $\exists x\neg\phi$ $\exists i$ 3

5. $\perp$ $\neg e$ 2,4

   ———-Box B3 closes———

6. $\phi[x_0/x]$ PBC 3-5

   ———-Box B2 closes———

7. $\forall x\phi$ $\forall$ i Box B2

8. $\perp$ $\neg e$ 1,7

   ———-Box B1 closes———

9. $\exists x\neg\phi$ PBC 2-8

(b) $\forall$ Introduction Rule

$$\frac{\Gamma \vdash \phi[x_0/x]}{\Gamma \vdash \forall x\phi} \quad \text{where } x \text{ does not occur free in } \Gamma.$$

$\forall$ Elimination Rule

$$\frac{\Gamma \vdash \forall x\phi}{\Gamma \vdash \phi[t/x]}$$

(c)

1. $\forall x(\phi \wedge \psi) \vdash \forall x(\phi \wedge \psi)$ Axiom
2. $\forall x(\phi \wedge \psi) \vdash \phi \wedge \psi$ $\forall e$ 1
3. $\forall x(\phi \wedge \psi) \vdash \phi$ $\wedge e1$ 2
4. $\forall x(\phi \wedge \psi) \vdash \forall x \phi$ $\forall i$ 3 (x is not free in $\forall x(\phi \wedge \psi)$)
5. $\forall x(\phi \wedge \psi) \vdash \psi$ $\wedge e2$ 2
6. $\forall x(\phi \wedge \psi) \vdash \forall x \psi$ $\forall i$ 5 (x is not free in $\forall x(\phi \wedge \psi)$)
7. $\forall x(\phi \wedge \psi) \vdash \forall x \phi \wedge \forall x \psi$ $\wedge i$ 4,6

Q3(a) Note that 1 is not in the vocabulary of our structure. So $\phi(x, y)$ can not be taken as $y = x + 1$.

Following are some of the correct answers.

1. $\phi(x, y) \equiv x < y \wedge \neg \exists z(x < z \wedge z < y)$
2. $\phi(x, y) \equiv \exists z[y = x + z \wedge z \neq 0 \wedge z \cdot z = z]$

(b) $\exists u \exists v[\forall a \forall b \forall c(1 \leq a < x \wedge \beta(u, v, a, b) \wedge \beta(u, v, a, c) \rightarrow b = c)$
$\wedge \beta(u, v, 1, 2)$
$\wedge\ \forall w \forall z_1(1 \leq w < x \wedge \beta(u, v, w, z_1) \rightarrow \beta(u, v, w + 1, 2 \cdot z_1))$
$\wedge \beta(u, v, x, y)]$

The above formula essentially asserts that there is a sequence $2, 2^2, \ldots, 2^i, 2^{i+1}, \ldots, 2^x$ whose x^{th} element is y.

Q4(a)

(i) Given below is a ND proof of this sequent

——-Box B1 begins———

1. $\exists y \forall x A(x, y)$ Assumption

——-Box B2begins———

x_0

——-Box B3 begins———

2. y_0 $\forall x A(x, y_0)$ Assumption
3. $A(x_0, y_0)$ $\forall e$, 2
4. $\exists y A(x_0, y)$ $\exists i$, 3

——-Box B3 closes———

5. $\exists y A(x_0, y)$ $\exists e$ 1, 2-4

——-Box B2 closes———

6. $\forall x \exists y A(x, y)$ $\forall i$ 2-5 (Box B2)

——-Box B1 closes———

7. $\exists y \forall x A(x, y) \rightarrow \forall x \exists y A(x, y)$ $\rightarrow i$ 1-6

(ii) Consider interpretation (N, A^N) where
$N = \{1, 2, 3, \ldots\}$ and $A^N(x, y) \equiv x < y$.
$\forall x \exists y A(x, y)$ is true.
($\because$ for every number there is a number bigger than it)
$\exists y \forall x A(x, y)$ is false.
($\because$ there is no number which is bigger than every number)
So $\forall x \exists y A(x, y) \rightarrow \exists y \forall x A(x, y)$ is false in this interpretation.

(b) $\Longrightarrow$ **(Left to right direction)**

Let M be a structure over Σ satisfying θ. We denote domain of M by $dom(M)$.

$M \models \theta$

$\Rightarrow$ for every $a, b \in dom(M)$ there is a $c \in dom(M)$ s.t. $M \models \phi(a, b, c)$.

We define $f^M(a, b) = c$, for some c as above.

$\Rightarrow M \models \phi(a, b, f^M(a, b))$

$\Rightarrow M$ extended with f^M is a model of χ.

$\Rightarrow \chi$ is satisfiable.

$\Longleftarrow$ **(Right to left direction)**

Assume χ is satisfiable in some structure M over $\Sigma \cup \{f\}$.
So for any $a, b \in dom(M)$, $M \models \phi(a, b, f^M(a, b))$
$\Rightarrow M \models \exists z \phi(a, b, z)$
Let M^- be the structure obtained by removing f^M. M^- is a structure over Σ.
$\Rightarrow M^- \models \exists z \phi(a, b, z)$ ($\because f$ does not occcur in ϕ)
$\Rightarrow M^- \models \forall x \forall y \exists z \phi(x, y, z)$
$\Rightarrow \theta$ is satisfiable.

Q5(a) Let $\theta \equiv (B \to \phi_1) \wedge (\neg B \to \phi_2)$. The desired proof is written below in linear form.

1. $(\phi_1)C_1(\psi)$ Premise
2. $(\phi_2)C_2(\psi)$ Premise
3. $\theta \wedge B \to \phi_1$ Tautology
4. $(\theta \wedge B)C_1(\psi)$ Implied 3,1
5. $\theta \wedge \neg B \to \phi_2$ Tautology
6. $(\theta \wedge \neg B)C_2(\psi)$ Implied 5,2
7. $(\theta)\mathbf{if}\,B\{C_1\}\mathbf{else}\{C_2\}(\psi)$ **if**-Statement 4,6

(b) **(i)** Let $div(i, j)$ stand for 'i divides j'.

Define $inv(k) \equiv \forall j(0 \leq j < k \wedge div(3, j) \to a[j] \neq x)$

$\psi \equiv (i_0 < n \to \alpha) \wedge (i_0 = n \to inv(n))$,

where $\alpha \equiv (a[i_0] = x) \wedge div(3, i_0) \wedge inv(i_0)$

(ii) Loop invariant: $div(3, i) \wedge inv(i)$

Annotated Program is given below with assertions shown between two asterisks.

$*\text{true}*$

$\Downarrow$

$*div(3, 0) \wedge inv(0)*$

$i = 0;$

$*div(3, i) \wedge inv(i)*$

while $(a[i] \neq x \wedge i + 3 < n)$ {

$*div(3, i) \wedge inv(i) \wedge (a[i] \neq x \wedge i + 3 < n)*$

$\Downarrow$

$*div(3, i + 3) \wedge inv(i + 3)*$

$i = i + 3;$

$*div(3, i) \wedge inv(i)*$

}

$*div(3, i) \wedge inv(i) \wedge (a[i] = x \vee i + 3 \geq n)*$

$\Downarrow$

$*(a[i] = x \to \theta) \wedge (a[i] \neq x \to inv(n))*$

if $(a[i] == x)$ $*\theta*\{i_0 = i;\}$

else $*inv(n)*\{i_0 = n;\}$

$*\psi*$

where $\theta \equiv (i < n \to a[i] = x \wedge div(3, i) \wedge inv(i)) \wedge (i = n \to inv(n))$

For **'if ...else ...'** command we have used rule in part (a).

We need to prove two implications shown as $\Downarrow$ above. Namely:

1. $div(3, i) \wedge inv(i) \wedge (a[i] \neq x \wedge i + 3 < n)$ implies $div(3, i + 3) \wedge inv(i + 3)$
2. $div(3, i) \wedge inv(i) \wedge (a[i] = x \vee i + 3 \geq n)$ implies $(a[i] = x \to \theta) \wedge (a[i] \neq x \to inv(n))$

Proof of (1):

$div(3, i)$ clearly implies $div(3, i+3)$.

We only need to show $inv(i+3)$.

That is, for any $j < i+3$ s.t. $div(3, j)$ we need to show that $a[j] \neq x$.

$div(3, i) \wedge j < i+3 \wedge div(3, j)$ implies $j \leq i$.

For $j < i$ and $div(3, j)$, $a[j] \neq x$ follows from $inv(i)$.

For $j = i$, $a[i] \neq x$ is given.

This shows $inv(i+3)$.

Proof of (2):

We consider two cases:

1. $a[i] = x$.
   In this case, first conjunct of θ is immediate, as everything on the right of implication is true.
   For the second conjunct of θ, if $i = n$ then $inv(n)$ is $inv(i)$ but $inv(i)$ is given. This shows θ.
   The implication starting with $a[i] \neq x$ trivially holds as we are considering the case $a[i] = x$.
2. $a[i] \neq x \wedge i+3 \geq n$.
   The conjunct $(a[i] = x \rightarrow \theta)$ holds trivially.
   We only need to show $inv(n)$.
   In the proof of (1) above, we have shown $inv(i+3)$ from assumption$div(3, i) \wedge inv(i) \wedge a[i] \neq x$.
   So $inv(i+3)$ holds in the present case also.
   As $n \leq i+3$, $inv(i+3)$ implies $inv(n)$.

Q6

(a) $$\frac{(\eta \wedge B \wedge 0 \leq E = E_0) C (\eta \wedge 0 \leq E < E_0)}{(\eta \wedge 0 \leq E) \textbf{while} B\{C\} (\eta \wedge \neg B)}$$

(b) $\phi \equiv x \geq 0 \wedge even(x)$

Annotated Program is

$*x \geq 0 \wedge even(x)*$

$\textbf{while}(x \neq 0)\{$

$*even(x) \wedge x \neq 0 \wedge 0 \leq x = E_0*$

$\Downarrow$

$*even(x-2) \wedge 0 \leq x-2 < E_0*$

$x = x - 2;$

$*even(x) \wedge 0 \leq x < E_0*$

$\}$

$*even(x) \wedge x = 0*$

$\Downarrow$

(true)

In above annotation we have used loop invariant $even(x)$ and for termination expression E is x.

Proof of implications:

1. Last implication '$even(x) \wedge x = 0$ implies $true$' is obvious.
2. We only have to show that
   $even(x) \wedge x \neq 0 \wedge 0 \leq x = E_0$ implies
   $even(x-2) \wedge 0 \leq x-2 < E_0$.
   This follows from:
   (a) $x = E_0 \to x - 2 < E_0$
   (b) $x \neq 0 \wedge even(x) \wedge x \geq 0$ implies $x > 0 \wedge even(x)$
   which implies $x - 2 \geq 0 \wedge even(x-2)$.

——————————————x-x-x——————————————