CS202A Endsem Exam

Nov 20, 2015
Duration 3 hours

Closed Book
Maximum Marks 80

Instructions

- Write your answers neatly, to the point and with logical clarity.
- Begin each question from a new page. Answer all parts of a question together.
- Strike out all your rough work.

Q1

(a) Propositional connectives $\rightarrow$ and $\neg$ are expressively complete. In other words, every propositional formula is logically equivalent to a formula using connectives $\rightarrow$ and $\neg$ only. Express connective $\wedge$ in terms of connectives $\rightarrow$ and $\neg$ only.

(b) Show that there is no tautology which is logically equivalent to a formula constructed using logical connectives $\vee$ and $\wedge$ only.

(c) Propositional connectives $\wedge$ and $\neg$ are expressively complete. Consider ND system with rules for connectives $\wedge$, $\neg$ and $\perp$ only. Show that introduction and elemination rules for $\vee$ can be derived in this system where $A \vee B$ in these rules is replaced by $\neg(\neg A \wedge \neg B)$.

(Marks 2+2+8)

Q2

(a) Give a ND proof of $\neg\forall x\phi \vdash \exists x\neg\phi$

(b) We have seen in an assignment question a propositional proof system based on sequents instead of Natural Deduction. Every ND rule gives rise to a sequent rule. For each ND rule for $\forall$ quantifier, give the corresponding sequent rule. State clearly any condition on free variables in these rules.

(c) Give a derivation of $\forall x(\phi \wedge \psi) \vdash \forall x\phi \wedge \forall x\psi$ in sequent system of part **(b)**.

(Marks 8+3+3)

Q3 Let $N = (\{0,1,2,3\ldots\}, +, \cdot, <, 0)$ be the structure of all non-negative integers with standard interpretation of function, relation and constant symbols.

We say a FO formula $\phi(x,y)$, in vocabulary $\{+,\cdot,<,0\}$, defines successor function if for all σ,

$N, \sigma \models \phi(x,y)$ iff $\sigma(y) = \sigma(x) + 1$.

[In the following, we write condition in last line more intuitively as $N \models \phi(x,y)$ iff $y = x + 1$].

(a) Give a formula $\phi(x,y)$ as above.

(b) Let $\beta(x,y,i,z)$ be a formula in vocabulary $\{+,\cdot,<,0\}$ s.t. for any sequence $n_1,\ldots,n_k$ of numbers there exist $a,b \in N$ s.t. for all $i \in \{1,\ldots,k\}$,

$N \models \beta(a,b,i,z)$ iff $z = n_i$.

[This is the same as Godel's β predicate given in practice problems].

Assume that you are given β predicate. Using β predicate freely in your formula define a FO formula $\psi(x,y)$, s.t. $N \models \psi(x,y)$ iff $y = 2^x$.

(Marks 4+6)

Q4

(a) For each of the formula below state if it is valid or not. If it is valid give a ND proof of it. Otherwise give an interpretation falsifying it.

(i) $\exists y \forall x A(x,y) \to \forall x \exists y A(x,y)$

(ii) $\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)$

(b) Consider a sentence $\theta \equiv \forall x \forall y \exists z \phi(x,y,z)$ in vocabulary Σ, where $\phi(x,y,z)$ is a quantifier free formula. Let f be a binary function symbol s.t. $f \notin \Sigma$. Define sentence χ as $\chi \equiv \forall x \forall y \phi(x,y,f(x,y))$, over vocabulary $\Sigma \cup \{f\}$. Show that θ is staisfiable iff χ is satisfiable.

(Marks (4+4)+6)

Q5

(a) In Hoare logic proof rule for **if** command is as follows.

$$\frac{(B \wedge \phi) C_1 (\psi) \quad (\neg B \wedge \phi) C_2 (\psi)}{(\phi) \mathbf{if} B\{C_1\} \mathbf{else} \{C_2\} (\psi)}$$

Using this and other rules in Hoare logic derive the following rule.

$$\frac{(\phi_1) C_1 (\psi) \quad (\phi_2) C_2 (\psi)}{((B \to \phi_1) \wedge (\neg B \to \phi_2)) \mathbf{if} B\{C_1\} \mathbf{else} \{C_2\} (\psi)}$$

(b) Consider program P below.

$i = 0;$

while $(a[i] \neq x \wedge i + 3 < n)$ $\{$

$\quad i = i + 3;$

$\}$

if $(a[i] == x)$ $\{i_0 = i;\}$

$\quad$ **else**$\{i_0 = n;\}$

['a' is an array of length n in the above program.]

(i) For precondition $n \geq 1$, what is the strongest post condition ψ after execution of P?

Write ψ as $(i_0 < n \to \ldots) \wedge (i_0 = n \to \ldots)$, where spaces '...' have to be replaced by appropriate formulae.

(ii) Prove $\vdash_{par} (n \geq 1) P (\psi)$.

What is the loop invariant in your proof?

Mark various points in your program by appropriate valid formulae. Use abbreviations for long, repetitive formulae to avoid clutter. Do not use needlessly complex formulae. You may also use abbreviations for better readability.

For any non-obvious implication give a separate argument. Those implications which do not involve program phrases, can be proved by usual informal mathematical reasoning.

Apart from the correctness, marks will also depend on readability, neatness and logical clarity of your answer.

(Marks 4+(4+(3+5+6)))

Q6

(a) Write the total correctness rule for **while** programs in Hoare logic.

(b) Give a formula ϕ, true on the **largest** set of states s.t.

$\vdash_{tot}$ (ϕ) **while**$(x \neq 0)\, \{x = x - 2;\}\, (true)$ holds.

Also prove $\vdash_{tot}$ (ϕ) **while**$(x \neq 0)\, \{x = x - 2;\}\, (true)$ for your ϕ.

(Marks 2+(2+4))

————————————x-x-x————————————