Practice Problems

Q1

(a) Give a ND proof of $\forall x\phi(x) \to \psi \vdash \exists x(\phi(x) \to \psi)$, where $x \notin free(\psi)$.

(b) Convert the following formula in prenex normal form.

$\forall x\exists y(A(x) \to B(y)) \to \exists y\forall x(A(x) \to B(y))$

Q2 Consider the following formulae.

1. $\forall x\neg R(x,x)$
2. $\forall x\forall y\forall z(R(x,y) \wedge R(y,z) \to R(x,z))$
3. $\forall x\exists y R(x,y)$

(a) Show a model of these formulae. (That is, show a structure (A, R^A), in which all these formulae are true).

(b) Argue clearly that evey model of these formulae has infinite domain.

Q3 Let $I = (M, \sigma)$ be an interpretation. We denote the value of any term r under this interpretation as $I(r)$.

Let $I' = (M, \sigma[I(r)/x])$.

(a) For any term t show that $I'(t) = I(t[r/x])$. [Hint: use induction on t]

(b) Using (a) show that $I \models \phi[r/x]$ iff $I' \models \phi$

(c) Using (b) prove soundness of proof system for FOL covered in class.

Q4 In this question we see the construction of Godel's β predicate which allows to code finite sequence of natural numbers (of unbounded length) in arithmetic.

Let $n_1, \ldots n_k$ be any sequence of natural numbers.

Let $m = max\,\{k, n_1, \ldots n_k\}$.

Define for $i \in \{1, \ldots, k\}$, $p_i = 1 + (i+1)(m!)$.

(a) Show that p_i, p_j are relatively coprime for $i \neq j$.

(b) Define a first order formula $\beta(x, y, i, z)$ over the vocabulary of natural number structure ($\{+, \times, 0, <\}$), s.t. for any sequence $n_1, \ldots n_k$ of natural numbers there exist x and y s.t. for all $1 \leq i \leq k$, $N \models \beta(x, y, i, z)$ iff $z = n_i$. [Hint: use CRT, y can be taken as $m!$ above].

Q5 Consider the program below which we denote as P in the following.

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y=0;

while (y*y <= x){

y=y+1;

}

y=y-1;
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Prove the following in Hoare logic. The proof should be presented by inserting appropriate assertions in the program and justifying any non-obvious implications separately.

(a) $\vdash_{par} (x \geq 1)\ P\ (y^2 \leq x \wedge x < (y+1)^2)$

(b) $\vdash_{tot} (x \geq 1)\ P\ (y^2 \leq x \wedge x < (y+1)^2)$

(c) $\vdash_{par} (x = 0)\ P\ (y = 0)$

(d) $\vdash_{tot} (x < 0)\ P\ (x < 0)$.

——————————x-x-x——————————