

Lecture 3: Counting

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Counting problems arise in almost every aspect of computer science. In this lecture we will learn some basic techniques and principles for counting.

1 Basic counting

There are two very simple rules used extensively to count.

1. Sum rule: We need to choose one element from two sets, first containing a elements and the other b . If two sets are *disjoint* then it can be done in $a + b$ ways.
2. Product rule: We need to choose one element *each* from two sets, first one with a elements and the other b . Then total number of ways is ab .

We have taken only two sets, but the rule can be generalized to multiple sets easily. Lets do some examples.

Example 1. How many bit strings are there of length 7?

There are 7 positions. In each of these positions, there are two possibilities. Applying product rule, there are 128 possibilities.

Exercise 1. How many numbers are there between 1000 and 9999 (including both) which are divisible by 3?

Example 2. How many palindromes (words) are there of length n ?

If n is even then $26^{\frac{n}{2}}$. If n is odd then $26^{\frac{n+1}{2}}$.

Example 3. How many ways are there to put m balls into n bins. Assume $m \leq n$.

1. Balls are distinct and bins are distinct. Every ball has n choices. Hence n^m .

Exercise 2. Why is the answer not m^n by looking at the opposite argument.

2. Balls are not distinct but bins are distinct. Take m identical balls and $n - 1$ identical sticks and permute them. Every permutation gives a different arrangement. So there are $\binom{m+n-1}{m}$ ways.
3. Both are indistinguishable. Convince yourself that this is equal to number of partitions of m . We don't have a closed form formula for this number.
4. Balls are distinct but bins are not. This is again a difficult problem. Look at *Bell's number* for more information.

Exercise 3. Where did we use $m \leq n$.

1.1 Double counting

Lets consider the following problem. Show that in a conference, the number of members who shake hands an odd number of time is even.

We will count the pairs (m_i, m_j) , say P , where member m_i shook hands with m_j . We know that P is twice the number of total handshakes. Hence P is even.

Counting P another way, it is the sum of handshakes done by each member. If the number of members who shook hands an odd number of time is odd then P will be odd (why?). But since P is even, it implies, the number of members who shook hands an odd number of time is even.

This trick is called *double counting*. The idea is to count one particular quantity in two different ways. Since we know the both counting should give identical results, we can derive many relations between them.

* These notes are prepared with the help of Combinatorics book (Peter Cameron)

2 Inclusion exclusion

Suppose we have 100 students in an institute using various social media websites. Say 44 use Facebook, 50 use Twitter and 56 use Google plus. It is also known that how many of them use multiple websites. 27 use Facebook and Twitter, 31 are on Facebook and Google, also 34 use Google and Twitter. There are 19 who use all three websites. How many students are there who do not use any website?

This kind of question can be visualized easily using a Venn-diagram 2.

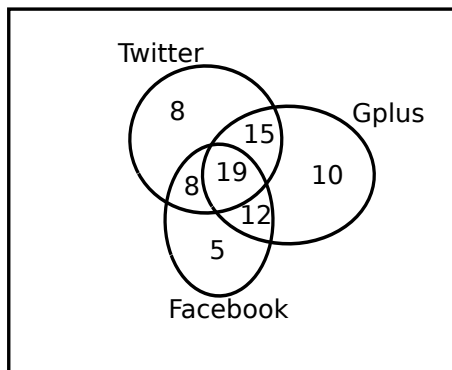


Fig. 1. Website Users

This Venn-diagram can be easily drawn using the data above. Looking at the diagram, it is quite clear that 23 students do not use any social media website. But if there were twenty websites (lot of websites), this strategy is not feasible.

We try to count the number of students using at least one website in a different way. Our first guess would be to sum up students in using different websites ($44 + 50 + 56 = 150$). But clearly this counts student using two websites twice, so lets subtract the students who are in the intersection of two websites.

So the next guess would be $150 - (27 + 31 + 34) = 58$. Again, students using all three websites were counted 3 times in the beginning, then subtracted thrice, so was not counted at all. Hence the true count is $58 + 19 = 77$. So there are 23 students who do not use any of the three websites.

This argument can be generalized for more than three websites. Suppose there is a universe U with subsets $A_1, A_2, \dots, A_n$. For our previous example, U will be the set of all students ($|U| = 100$) and A_1 will be the set of students using facebook and so on.

We are given their intersections $A_I = \cap_{i \in I} A_i, I \subseteq [n]$. Then the number of elements not in any of the sets A_i is given by,

$$|U - \cup_{i \in [n]} A_i| = \sum_{I \in [n]} (-1)^{|I|} |A_I|.$$

Note 1. $A_\emptyset$ is the universe U itself. Why?

This is called the principle of inclusion and exclusion.

Exercise 4. Why is it called inclusion-exclusion?

Proof. We will show the equality by finding out how many times do we count an element of U . If an element $u \in U$ is not contained in any of the sets A_i then it will be counted exactly once by the term $A_\emptyset$.

So we only need to show that every other element is counted 0 times overall. Suppose an element u is contained in A_j 's for every $j \in J$, where J is non-empty. Then it will be counted,

$$c = \sum_{I \subseteq J} (-1)^{|I|},$$

times. Suppose $|J| = k$, then

Exercise 5. show that $c = \sum_{i=0}^k (-1)^i \binom{k}{i} = 0$.

This proves that every element of U is counted exactly once if it is not in any of A_i 's and not counted otherwise. Hence proved. $\square$

Example 4. Derangements: Suppose we have n letters and n envelopes with one envelope marked for one particular letter. In how many ways could you place letter in envelopes (one letter goes to exactly one envelope), s.t., no letter goes to the correct envelope?

This is the standard application of inclusion-exclusion and known as *derangements*. There are $n!$ ways to put letters into envelopes and that is our universe U . Suppose A_i is the set of ways when letter i goes to its correct envelope. Hence, we are interested in $|U - \cup_{i \in [n]} A_i|$.

To apply the inclusion-exclusion formula, we need to calculate A_I . After placing $|I|$ letters in the correct position, we have $(n - |I|)!$ ways to place remaining letters. There are $\binom{n}{i}$ subsets of size i . Then the number of derangements are,

$$\sum_{i=0}^n (-1)^i \binom{n}{i} (n - i)! = n! \sum_{i=0}^n \frac{(-1)^i}{i!}.$$

3 Recurrence relations

Recursion is a very helpful tool to solve problems in computer science and mathematics. We will look at recursion as an aid to counting.

Suppose rabbit population needs to be introduced to an island. The plan is to put a pairs of new-born rabbits in the first month and then b pairs in the second month. The pairs start to reproduce 2 months after their birth. Each pair produces one offspring every month. What will be the population after n months (assume that rabbits don't die)?

Say we call the population in the n^{th} month, F_n . There will be two kind of rabbits making up F_n : new-born and already present. F_{n-1} would be already present and F_{n-2} will be new-borns. So,

$$F_n = F_{n-1} + F_{n-2}.$$

This is called a recurrence relation for F_n . We gave a *combinatorial argument* for its proof.

With the initial condition ($F_0 = a, F_1 = b$), this recurrence gives us an easy algorithmic way to compute the population in the month n . Many a times, even if it is hard to come up with an explicit formula for a mathematical quantity, recurrence relation might give us some information about it.

Exercise 6. Suppose $a = 1, b = 1$. Show that,

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

The case $a = 1, b = 1$ is very special and known as *Fibonacci sequence*. That is, the numbers F_n are called Fibonacci sequence if they satisfy recurrence $F_n = F_{n-1} + F_{n-2}$ with the initial condition $F_1 = F_0 = 1$.

Exercise 7. Write the recurrence relation for S_n where S_n is the number of subsets of a set with n elements.

3.1 Generating functions

The power-series $\phi(t) = \sum_{i \geq 0} F_i t^i$ is called the *generating function* for sequence F_i .

Exercise 8. Why is it not called a polynomial but a power-series?

We will be doing additions, multiplications and other operations on these power-series without worrying about the notion of convergence. These are known as *formal power series*. The justification for not worrying about the convergence is outside the scope of this course.

What can we do with generating function? Say we consider the Fibonacci sequence. If we multiply it with t , we get,

$$t\phi(t) = \sum_{i \geq 0} F_i t^{i+1} = \sum_{i \geq 1} F_{i-1} t^i.$$

Notice how we changed the summation index. Convince yourself that it works. Multiplying it again by t ,

$$t^2\phi(t) = \sum_{i \geq 0} F_i t^{i+2} = \sum_{i \geq 2} F_{i-2} t^i.$$

Since $F_{i-1} + F_{i-2} = F_i$, adding above two equations will give us the original generating function. Almost (why?) !!

$$\phi(t) = t\phi(t) + t^2\phi(t) + 1$$

All the coefficients for terms higher than t^2 agree. The constant term 1 appears by comparing the constant coefficient and the coefficient of t . This is where we use initial conditions !! We get the formula for $\phi(t)$,

$$\phi(t) = \frac{1}{1 - t - t^2}.$$

We can factorize the polynomial, $1 - t - t^2 = (1 - \alpha t)(1 - \beta t)$.

Exercise 9. Show that $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.

We can simplify the formula a bit more.

$$\phi(t) = \frac{1}{1 - t - t^2} = \frac{c_1}{1 - \alpha t} + \frac{c_2}{1 - \beta t} = c_1(1 + \alpha t + \alpha^2 t^2 + \dots) + c_2(1 + \beta t + \beta^2 t^2 + \dots)$$

We got the second equality by putting $c_1 = \frac{1}{\sqrt{5}}\alpha$ and $c_2 = -\frac{1}{\sqrt{5}}\beta$. The final expression gives us an explicit formula for the Fibonacci sequence.

$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

This is actually a very strong method and can be used for solving *linear recurrences* of the kind, $S_n = a_1 S_{n-1} + \dots + a_k S_{n-k}$. Here $a_1, a_2, \dots, a_k$ are constants. Suppose $\phi(t)$ is the recurrence for S_n , then by the above method,

$$\phi(t) = \frac{b_1 + b_2 t + \dots + b_k t^{k-1}}{1 - a_1 t - a_2 t^2 - \dots - a_k t^k} = \frac{c_1}{1 - \alpha_1 t} + \dots + \frac{c_k}{1 - \alpha_k t}.$$

Where $\alpha_1, \dots, \alpha_k$ are the roots of polynomial $x^k - a_1 x^{k-1} - \dots - a_k = 0$ (replace t by $\frac{1}{x}$). This is known as the *characteristic polynomial* of the recurrence.

Exercise 10. How are the coefficients $b_1, \dots, b_k$ or $c_1, \dots, c_k$ determined?

So we get $S_n = c_1\alpha_1^n + \dots + c_k\alpha_k^n$.

Note 2. For a linear recurrence if F_n and G_n are solutions then their linear combinations $aF_n + bG_n$ are also solutions. For a k term recurrence, the possible solutions are $\alpha_1^n, \dots, \alpha_k^n$ and their linear combinations (where $\alpha_1, \dots, \alpha_k$ are roots of the characteristic polynomial). The coefficients of the linear combination are fixed by the initial conditions.

3.2 Exponential generating function

A permutation is called an *involution* if all cycles in the permutation are of length 1 or 2. We are interested in counting the total number of involutions of $\{1, 2, \dots, n\}$, call that $I(n)$.

There can be two cases.

1. The number n maps to itself. This case will give rise to $I(n-1)$ involutions.
2. The number n maps to another number i . There are $n-1$ choices of i and then we can pick an involution for remaining $n-2$ numbers in $I(n-2)$ ways.

By this argument, we get a simple recurrence,

$$I(n) = I(n-1) + (n-1)I(n-2).$$

Exercise 11. Why is this not a linear recurrence?

Since the coefficient of $I(n-2)$ is not a constant, we can't apply the usual approach of generating functions. Even without getting an explicit formula for $I(n)$, the recurrence can give us some information about the quantity.

Theorem 1. For $n \geq 2$, the number $I(n)$ is even and greater than $\sqrt{n!}$.

Proof. Both the statements can be proven using induction. Notice that $I(2) = 2$ and $I(1) = 1$.

Exercise 12. What will be the base case for proving that $I(n)$ is even?

For the second part, $I(n) \geq \sqrt{n!}$, notice that,

$$1 + \sqrt{n-1} \geq \sqrt{n}.$$

□

In the case of involutions, the regular generating function will not be of much help. We define *exponential generating function* for a sequence $I(n)$ to be,

$$\theta(t) = \sum_{k \geq 0} \frac{I(k)t^k}{k!}.$$

Exercise 13. Why is it called exponential generating function?

We can actually come up with a closed form solution for the exponential generating function of involutions. We will differentiate the function $\theta(t)$,

$$\frac{d}{dt}\theta(t) = \sum_{k \geq 1} \frac{I(k)t^{k-1}}{(k-1)!} \quad \text{(formal differentiation)} \quad (1)$$

$$= \sum_{k \geq 1} \frac{I(k-1)t^{k-1}}{(k-1)!} + \sum_{k \geq 1} \frac{(k-1)I(k-2)t^{k-1}}{(k-1)!} \quad \text{(recurrence relation)} \quad (2)$$

$$= \theta(t) + t \sum_{k \geq 2} \frac{I(k-2)t^{k-2}}{(k-2)!} \quad \text{(second term's first entry is zero)} \quad (3)$$

$$= \theta(t) + t\theta(t). \quad (4)$$

This transforms into the differential equation,

$$\frac{d}{dt} \log \theta(t) = 1 + t$$

We can solve this,

$$\theta(t) = e^{t + \frac{t^2}{2}} + c.$$

Comparing the constant coefficient, we get $c = 0$ (since $I(0) = 1$),

$$\theta(t) = e^{t + \frac{t^2}{2}}.$$

Note 3. Again we should notice that the power series we are considering are not shown to be well behaved (convergence etc.). But there is a justification for being able to differentiate and do other operations on them, but it is outside the scope of this course.

Exercise 14. The number of partitions of a set with n elements is called the *Bell number*, B_n . Show that it satisfies the recurrence,

$$B_n = \sum_{i=1}^n \binom{n-1}{i-1} B_{n-i}.$$

This is also equal to the number of equivalence relations on a set with n elements. This is the case of distinct balls but indistinguishable bins.

4 Pigeonhole principle

This is one of the simplest principles in mathematics which has numerous applications.

Theorem 2. *If there are $n + 1$ pigeons and n pigeonholes then at least one pigeonhole will have more than 1 pigeon.*

This seemingly obvious theorem has many nice applications. Lets take a look at few of them. To start with, it implies, in a group of 367 friends there are at least two whose birthday coincides. Here friends are the pigeons and birth-date are the pigeonholes.

Example 5. Lets say that there are $n \geq 2$ users of Facebook. Show that there exist at least two people who have same number of friends. We can assume that $n \geq 3$. For small cases you can easily check that the theorem holds. Clearly the number of friends a user can have will only range from $\{0, 1, 2, \dots, n-1\}$. There are n people and n possible number of friends, so we can't apply the pigeonhole principle.

But if someone is friends with $n-1$ people then everyone has at least one friend. That means both 0 and $n-1$ can't appear in the possible number of friends. So there are n pigeons (users) and $n-1$ pigeonholes (number of friends), so at least two people will have same number of friends.

Exercise 15. Suppose there is an island in the shape of an equilateral triangle with side 2 kms. Is it possible to assign spaces for five houses on the island, such that, no two houses are within a distance of 1 km.

Let us consider the question we asked in the first class.

Example 6. For any n there exist at least one string of 0, 1 which is divisible by n .

The difficulty lies in figuring out what are pigeons and what are pigeonholes. Suppose we consider all possible strings of 0, 1. What remainder will they leave when divided by n ?

The remainder can only be $\{0, 1, \dots, n-1\}$. So we can take any $n+1$ strings, at least 2 of them will have the same remainder when divided by n . This only implies that there are two 0, 1 strings whose difference is divisible by n . But the difference need not be a 0, 1 string.

The question is then, can we have $n+1$ strings of 0, 1 whose difference is also an 0, 1 string? Consider the all 1's strings. There are more than $n+1$ such strings and also if we subtract smaller one from the bigger one, we will get an 0, 1 string.

Note 4. The string will actually be very special. It will be all 1's and then all 0's

Exercise 16. Can you extend the argument to show that there are infinitely many strings of 0, 1 divisible by n .

The pigeonhole principal can be extended slightly and again the proof is obvious (show it).

Theorem 3. *If there are $rn + 1$ pigeons and n pigeonholes then at least one pigeonhole will have more than r pigeons.*

Example 7. Given 6 vertices of a hexagon, join all pairs of vertices by either red or blue edge. Show that there is at least one monochromatic triangle (all edges of the same color).

Choose a vertex v . There are 5 edges going from it. Since there are 2 colors, at least three edges are of the same color by the new pigeonhole principle. Suppose these 3 edges are of red color and are v_1, v_2, v_3 .

There are 3 edges between v_1, v_2, v_3 . If all of them are blue then we have a monochromatic blue triangle (v_1, v_2, v_3) . Otherwise say v_1, v_2 is red then v, v_1, v_2 triangle is all red. Hence proved.

Exercise 17. Color the edges of a pentagon such that there is no monochromatic triangle.

Exercise 18. Read about Ramsey numbers.

5 Assignment

Exercise 19. Write pseudocode to find the next natural number in base b .

Exercise 20. A function with boolean input and output takes some arguments (True or False) and return a value True or False. Find out the total number of such functions with n arguments.

Exercise 21. Learn what a graph is. Then show that the sum of degree of each vertex is twice the number of edges.

Exercise 22. Show that the number of surjective (onto) mappings from a set with n elements to a set with k elements is,

$$\sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

Exercise 23. Read about Fibonacci numbers on the internet.

Exercise 24. Give a combinatorial argument for the recurrence,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Exercise 25. Find all possible solutions for a sequence S_n which satisfies,

$$S_n = S_{n-1} + 6S_{n-2}.$$

Exercise 26. Find the exponential generating function for Bell number B_n .

Exercise 27. Suppose α is an irrational. Show that for any n , there is a rational number $\frac{p}{q}$ with $q \leq n$, s.t.,

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{nq}.$$

Exercise 28. Show that if you choose any $n + 1$ numbers from the set $\{1, 2, \dots, 2n\}$. There will be at least two elements such that one divides the other.

References

1. P. J. Cameron. Combinatorics: Topics, Techniques and Algorithms. *Cambridge University Press*, 1994.
2. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.